M382D Second Exam
May 7, 2004

This exam consists of five problems. Do any THREE. Make it VERY clear which three you are attempting, or I may have to pick which problems to grade at random. All answers need to be justified. "Does there exist ..." really means "Prove the existence or nonexistence of ..."

Good luck!

Problem 1: Let $[y_0; y_1; y_2]$ be homogeneous coordinates on $Y = CP^2$, where y_i is a *complex* number (and hence should be thought of as two real numbers). Likewise, let $[x_0; x_1]$ be homogeneous coordinates on CP^1. Let $f : CP^1 \to CP^2$ be given by $f[x_0; x_1] = [x_0; 0; x_1]$, and let $g[x_0; x_1] = [x_0^2; x_0x_1; x_1^2]$. Let Z (called the hyperplane) be the image of f, and let S (a conic) be the image of g.

a) Is S transversal to Z?

Yes. $S \cap Z$ consists of 2 points, $[1; 0; 0]$ and $[0; 0; 1]$. Near $[1; 0; 0]$ use (complex) coordinates y_1/y_0 and y_2/y_0. The tangent space of Z allows arbitrary variations in the second coordinate but not the first. The tangent space of S allows variations in the first but not the second. Their direct sum is the tangent space of Y. The situation at $[0; 0; 1]$ is similar, with coordinates y_0/y_2 and y_1/y_2.

b) With the standard orientations (described below), compute $I(S, Z)$.

The previous argument shows that $g_*TX + TZ = TY$ (and not $-TY$), so both local intersection numbers are +1, so $I(S, Z) = +2$.

Locally, CP^n looks like C^n, which looks like $C^1 \times \cdots \times C^1$. $C^1 = R^2$ has the orientation e_x, e_y. C^n gets its orientation from C^1, so if $z_j = x_j + iy_j$, an oriented basis of C^n is $e_{x_1}, e_{y_1}, \ldots, e_{x_n}, e_{y_n}$. CP^n then gets its orientation from C^n.

Problem 2: Consider the genus-g torus X_g, with $(g \geq 0)$, embedded symmetrically in R^3, as in the figure on the blackboard. Let $f : X_g \to X_g$ be rotation by 180 degrees about the vertical axis.

a) Compute the Lefschetz number $L(f)$ for each genus.

When g is odd there are no fixed points, so $L(f) = 0$. When g is even there are two fixed points, where the vertical axis cuts into (and out of) the torus. At each fixed point df is a rotation, hence has eigenvalues -1 and -1, so the local Lefschetz number is +1, so $L(f) = 2$.

b) For which values of g is f homotopic to the identity map? Give a complete argument.

If $g > 1$, then the Euler characteristic is negative but $L(f)$ is non-negative, so f cannot be homotopic to the identity. If $g = 0$ or $g = 1$, then f IS homotopic to the identity, by the homotopy f_t = rotation by πt.

Problem 3: Construct the following, or prove they cannot exist:

a) A smooth vector field on S^4 with no zeroes.

Does not exist, since S^4 has nonzero Euler characteristic. (If a nonvanishing vector field existed, then the Hopf degree formula would imply $\chi = 0$.)

b) A smooth vector field on S^3 with no zeroes.

Exists. Thinking of S^3 as sitting in R^4, take $v(x_1, x_2, x_3, x_4) = (x_2, -x_1, x_4, -x_3)$.

c) A smooth vector field on S^1 with exactly one zero.

Exists. Take $v = (1 - \cos(\theta))\partial/\partial\theta$. The index of the zero happens to be zero.

Problem 4: Let S^2 be the 2-sphere and let T^2 be the standard (genus one) torus.

a) Does there exist a smooth degree 1 map $S^2 \to T^2$? (You may need to use covering spaces to do this).

No. Since S^2 is simply connected, any map from S^2 to T^2 lift to a map $S^2 \to R^2$, which is homotopic to a constant map. Projecting back down to T^2, we see that the original map is homotopic to a constant, and hence has degree zero.

b) Does there exist a smooth degree 1 map $T^2 \to S^2$.

Yes. Embed T^2 in R^3 in the usual way, and consider any point p not in the image of T^2. Now consider the winding map $f(x) = (x-p)/|x-p|$, which maps T^2 to S^2. If p is on the inside of T^2, this will have degree 1, while if it is outside the map will have degree zero.

Problem 5: On $\mathbb{R}^3$, let $\alpha = z dx \wedge dy + x dy \wedge dz - y dx \wedge dz$.

a)Take coordinates: $f : (0, \pi) \times (0, 2\pi) \to S^2 \subset R^3, f(\theta, \phi) = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$. Compute $f^*(dx)$, $f^*(dy)$, and $f^*(dz)$.

$f^*x = \sin(\theta)\cos(\phi)$, so $f^*(dx) = d(f^*x) = \cos(\theta)\cos(\phi)d\theta - \sin(\theta)\sin(\phi)d\phi$. Likewise, $f^*y = \sin(\theta)\sin(\phi)$, so $f^*(dy) = d(f^*y) = \cos(\theta)\sin(\phi)d\theta + \sin(\theta)\cos(\phi)d\phi$ and $f^*z = \cos(\theta)$, so $f^*(dz) = d(f^*z) = -\sin(\theta)d\theta$.

b) Compute $f^*\alpha$ in terms of $d\theta$ and $d\phi$, and integrating this over $(0, \pi) \times (0, 2\pi)$ to get $\int_{S^2} \alpha$.

Multiplying things out we get that $f^*\alpha = \sin(\theta)d\theta d\phi$, whose integral is 4π.

c) Compute $d\alpha$ and use Stokes' theorem in R^3 to get $\int_{S^2} \alpha$. [Your answer should agree with that of part b!]

$d\alpha = 3dx \wedge dy \wedge dz$. Since $S^2 = \partial B^3$, where B^3 is the unit ball, we have that

$$\int_{S^2} \alpha = \int_{B^3} d\alpha = 3Vol(B^3) = 4\pi.$$