Name:

M382D Midterm Exam
March 12, 2004

Do THREE of the five problems below. Do NOT attempt more than three, or I'll choose arbitrarily which three to grade and you might not like my choice! Of your three problems, at least one must be 1 or 2, and at least one must be 4 or 5. That is, of the $\binom{5}{3=10}$ possibilities, all are OK except 1-2-3 and 3-4-5.

Problem 1: a) Suppose that X is an n-dimensional manifold, x is a point on X, and $f^1, \ldots, f^{n-1} : X \to \mathbb{R}$ are smooth functions such that the differentials $df^1_x, \ldots, df^{n-1}_x$ are linearly independent at x. Prove there is a function $f^n : X \to \mathbb{R}$ such that $f^1, \ldots, f^n$ is a local coordinate system in a neighborhood of x.

Problem 2: a) Suppose $f : X \to Y$ is a smooth map from a compact manifold X to a connected manifold Y. Assume that df_x is invertible for all $x \in X$. Prove that f is surjective.

b) Find a counterexample if X is not compact.

Problem 3: Let X be a smooth manifold and let $f : X \to \mathbb{R}^3$ be a smooth map.

a) Is there necessarily a point $z \in \mathbb{R}^3$ such that $f^{-1}(x)$ is a smooth submanifold of X?

b) Is there necessarily a vertical line ℓ in $\mathbb{R}^3$ such that $f^{-1}(\ell)$ is a smooth submanifold of X?

Problem 4: a) Suppose we have the usual situation for intersection theory (X compact, Z closed submanifold of Y, and $dim(X) + dim(Z) = dim(Y)$) and that $f : X \to Y$ is homotopic to a constant map. Show that $I_2(f, Z) = 0$.

b) Suppose that $Y = \mathbb{R}^N$, that we have the usual setup for intersection theory, and that $f : X \to Y$ is any smooth map. Show that $I_2(f, Z) = 0$.

Problem 5: Prove that there exists a complex number z such that $z^7 + \cos(|z|^2)(35z^3 + iz^2 - 894) = 0$. Don't handwave! If you claim that two maps are homotopic, show the homotopy explicitly.