M382D Second Exam
May 7, 2004

This exam consists of five problems. Do any THREE. Make it VERY clear which three you are attempting, or I may have to pick which problems to grade at random. All answers need to be justified. "Does there exist ..." really means "Prove the existence or nonexistence of ..."

Good luck!

Problem 1: Let $[y_0, y_1, y_2]$ be homogeneous coordinates on $Y = CP^2$, where y_i is a *complex* number (and hence should be thought of as two real numbers). Likewise, let $[x_0, x_1]$ be homogeneous coordinates on CP^1. Let $f : CP^1 \to CP^2$ be given by $f[x_0, x_1] = [x_0, 0, x_1]$, and let $g[x_0, x_1] = [x_0^2, x_0 x_1, x_1^2]$. Let Z (called the hyperplane) be the image of f, and let S (a conic) be the image of g.

a) Is S transversal to Z?

b) With the standard orientations (described below), compute $I(S, Z)$.

Locally, CP^n looks like C^n, which looks like $C^1 \times \cdots \times C^1$. $C^1 = R^2$ has the orientation e_x, e_y. C^n gets its orientation from C^1, so if $z_j = x_j + iy_j$, an oriented basis of C^n is $e_{x_1}, e_{y_1}, \ldots, e_{x_n}, e_{y_n}$. CP^n then gets its orientation from C^n.

Problem 2: Consider the genus-g torus X_g, with $(g \geq 0)$, embedded symmetrically in R^3, as in the figure on the blackboard. Let $f : X_g \to X_g$ be rotation by 180 degrees about the vertical axis.

a) Compute the Lefschetz number $L(f)$ for each genus.

b) For which values of g is f homotopic to the identity map? Give a complete argument.

Problem 3: Construct the following, or prove they cannot exist:

a) A smooth vector field on S^4 with no zeroes.

b) A smooth vector field on S^3 with no zeroes.

c) A smooth vector field on S^1 with exactly one zero.

Problem 4: Let S^2 be the 2-sphere and let T^2 be the standard (genus one) torus.

a) Does there exist a smooth degree 1 map $S^2 \to T^2$? (You may need to use covering spaces to do this).

b) Does there exist a smooth degree 1 map $T^2 \to S^2$.

Problem 5: On $\mathbb{R}^3$, let $\alpha = z dx \wedge dy + x dy \wedge dz - y dx \wedge dz$.

a)Take coordinates: $f : (0, \pi) \times (0, 2\pi) \to S^2 \subset R^3, f(\theta, \phi) = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$. Compute $f^*(dx)$, $f^*(dy)$, and $f^*(dz)$.

b) Compute $f^*\alpha$ in terms of $d\theta$ and $d\phi$, and integrating this over $(0, \pi) \times (0, 2\pi)$ to get $\int_{S^2} \alpha$.

c) Compute $d\alpha$ and use Stokes' theorem in R^3 to get $\int_{S^2} \alpha$. [Your answer should agree with that of part b!]