Lie Groups Solutions, Problem Set # 4

Section 2.5:

2: If F is $\mathbf{g}-stable$, then $X\mathbf{v} \in F$ for all $X \in \mathbf{g}$, $\mathbf{v} \in F$. Likewise, $X^2\mathbf{v} = X(X\mathbf{v}) \in F$, and by induction $X^n\mathbf{v} \in F$. Since F is a vector space, $\exp(X)\mathbf{v} = \sum X^n\mathbf{v}/n! \in F$, so $\exp(\mathbf{g})$ sends F to itself. Thus the group generated by $\exp(\mathbf{g})$ sends F to itself. Since G is connected, that is all of G.

Conversely, if F is G-stable and $X \in \mathbf{g}$, then $\exp(tX)\mathbf{v} \in F$ for all $\mathbf{v} \in F$. Taking a derivative with respect to t at $t = 0$ means that $X\mathbf{v} \in F$.

5: $SO(3)$ and $SU(2)$ are NOT complex, nor are $O(3)$ or $SL(2, R)$ or the Euclidean group acting on R^2. (Any complex group must have an even real dimension, so these 3-dimensional examples are easily eliminated). However, $SL(2, C)$ is complex, as is $SL(n, C)$, and as is $GL(n, C)$. The triangle groups of Example 6 are complex (if E is a complex vector space) as is the group of affine transformations when E is complex. Finally, the direct product of two complex groups is complex.

7: (a) Any path through the origin in G can be written uniquely as the product of a path in M and a path in N: $\gamma(t) = \alpha(t)\beta(t)$, and at $t = 0$ we have $d\gamma/dt = d\alpha/dt + d\beta/dt$. Thus $\mathbf{g} = \mathbf{m} + \mathbf{n}$. Since $M \cap N = 1$, $\mathbf{m} \cap \mathbf{n} = 0$, so $\mathbf{g} = \mathbf{m} \oplus \mathbf{n}$. Since M is a subgroup, $\mathbf{m}$ is a sub-algebra. Since N is a normal subgroup, $\mathbf{n}$ is an ideal.

(b) By Baker-Campbell-Haussdorff, $\exp(-X)\exp(X+Y) = \exp(Z)$, with Z given by a sum of brackets. Since $\mathbf{n}$ is an ideal, all terms of the brackets are in $\mathbf{n}$, so $Z \in \mathbf{n}$, and we can define $A(X)Y = Z$. Note the expression $\exp A(X)Y$ should be read as $\exp(A(X)Y)$, and not as $(\exp A(X))Y$.

(c) First note that, by Dynkin's formula, $\exp(X)\exp(tY) = \exp(W(t))$, where $W(t) - X \in \mathbf{n}$, by the same argument as above. When N is Abelian, we re-do the derivation of Dynkin's formula as follows: Let $\exp(W(t)) = \exp(X)\exp(tY)$. Then $de^W/dt = e^W Y$. However, $de^W/dt = e^W[(1 - \exp(-adW))/ad(W)]dW/dt$, so $dW/dt = [(1 - \exp(-adW))/ad(W)]^{-1}Y$. However, acting on $\mathbf{n}$, $ad(W) = ad(X)$, since $\mathbf{n}$ is Abelian. Thus $dW/dt = A(X)^{-1}Y$, so $W(1) = W(0) + A(X)^{-1}Y = X + A(X)^{-1}Y$. That is, we have proven that $\exp(X)\exp(Y) = \exp(X + A(X)^{-1}Y)$. Now, replacing Y with $A(X)Y$, we get $\exp(X)\exp(A(X)Y) = \exp(X + Y)$.

(d) The affine group is the set of all matrices $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with a invertible and $b \in E$. This is (uniquely) factored as $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}$.

11: Recall that we have an inner product on $\mathbf{g}$, namely $\langle X|Y\rangle = \mathrm{Tr}(X^*Y)$. Relative to this inner product, $(ad(Z))^* = ad(Z^*)$, where on both sides the superscript * means adjoint. This is easily checked: $\langle X|ad(Z)Y\rangle = \text{Tr}(X^*(ZY - YZ)) = \text{Tr}((X^*Z - ZX^*)Y) = \langle ad(Z^*)X|Y\rangle$.

(a) Since Z is self-adjoint, $ad(Z)$ is self adjoint, hence diagonalizable with real eigenvalues, so $\mathbf{g}$ is the direct sum of eigenspaces with real eigenvalues.

(b) If $X \in \mathbf{g}_\lambda$ and $Y \in \mathbf{g}_\mu$, then by Jacobi, $[Z, [X, Y]] = [[Z, X], Y] + [X, [Z, Y]] = [\lambda X, Y] + [X, \mu Y] = (\lambda + \mu)[X, Y]$.

(c) If $X \in \mathbf{g}_\lambda$, then $[Z, X^*] = ZX^* - X^*Z = (-Z^*X + XZ^*)^* = -[Z^*, X]^* = -[Z, X]^* = -\lambda X^*$.

(d) The fact that $\mathbf{q}$ is a subalgebra follows from (b). The fact that $\mathbf{k}$ is a subalgebra comes from the fact that $[X, Y]^* = -[X^*, Y^*]$. To see that $\mathbf{k} + \mathbf{q} = \mathbf{g}$ (not necessarily direct sum!), we decompose an arbitrary element of $\mathbf{g}$ into a $\mathbf{k}$ piece and a $\mathbf{q}$ piece. By (a), we can assume with loss of generality that $X \in \mathbf{g}_\lambda$. If $\lambda \geq 0$, then $X \in q$. If $\lambda < 0$, then $X = (X - X^*) + X^*$, with $X - X^* \in \mathbf{k}$ and $X^* \in \mathbf{q}$.

(e) (This is closely related to polar decomposition.) To see that $L(K) = \mathbf{k}$, note that the derivative of the equation $k(t)^*k(t) = 1$ at $t = 0$ is $X^* + X = 0$, where $X = dk/dt$. Thus all elements of $L(K)$ are anti-hermitian. Likewise, the exponential of any anti-hermitian elements of $\mathbf{g}$ are both unitary and in G, hence in K. It's obvious that $\exp(\mathbf{q}) \subset N_G(\mathbf{q})$, and hence that $\mathbf{q} \subset L(Q)$. Conversely, if $Y \in L(Q)$, then $\exp(Yt) \in Q$, so $\exp(Yt)X\exp(-Yt) \in \mathbf{q}$ for all $X \in \mathbf{q}$, so $[Y, X] \in \mathbf{q}$. But $Z \in \mathbf{q}$, so $[Y, Z] \in \mathbf{q}$. But this means that $Y \in \mathbf{q}$. To show that $G = KQ$, it suffices by problem 10 to show that KQ is closed. So suppose that we have a sequence $k_j q_j$ that converges (in G). Since K is compact, there is a subsequence such that k_j converges. But if k_j and $k_j q_j$ both converge, then so does $k_j^* k_j q_j = q_j$, and we have that $\lim k_j q_j = \lim k_j \lim q_j \in KQ$.

12: $K = SO(2)$, and Q is the group of upper-triangular matrices with determinant 1. (Called "B" in Lemma 3B of section 2.1).

Section 2.6:

For these problems, note that the adjoint action of a group on its Lie algebra preserves a bilinear form on the Lie algebra, namely $\langle X|Y\rangle = -Tr(XY)$. Call this form K (for Killing). The adjoint action Ad gives a homomorphism from G to $Aut(K)$. In each example it is easy to see that the infinitesimal action ad is 1–1. Since the groups have the same dimension (in these examples), this induces a covering.

6: Define an action of $SL(2, C)$ on C^3 as follows. First identify C^3 with the Lie algebra $sl(2, C)$, and then take the adjoint action of $SL(2, C)$ on $sl(2, C)$. That is, if $a \in SL(2, C)$ and $X \in sl(2, C)$, let $\rho(a)X = Ad(a)X = aXa^{-1}$. This is just the complexification of the adjoint action of $SU(2)$ on $su(2)$, hence is the complexification of the action of $SO(3)$ on R^3, hence is an action of $SO(3,C)$ on C^3.

7: (a) The bilinear form has signature (2,1), so the adjoint action gives a map $SL(2,R) \to SO(2,1)$. Since $SL(2,R)$ is connected, the image is connected, hence is in the identity component of $SO(2,1)$. Since it is 3-dimensional, it IS the identity component.

(b) The Lie algebra is spanned by $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, for which the Killing form has signature (1,2). So the image of the double cover map is a connected 3-dimensional subgroup of $SO(1,2) = SO(2,1)$, hence is the identity component.

(c) The Lie algebra of $SL(2,R)$ is spanned by the anti-hermitian matrix $X_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and the Hermitian matrices $X_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $X_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, while $su(1,1)$ is spanned by the anti-hermitian matrix $Y_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and the Hermitian matrices $Y_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $Y_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. These are conjugate by $P = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$. That is, $Y_k = PX_kP^{-1}$. This exponentiates to give $SU(1,1) = PSL(2,R)P^{-1}$.

8: (a) $SL(2,C) \times SL(2,C)$ acts on $M_2(C) = C^4$ (NOT C^2 – that's a typo) by $X \to aXb^{-1}$. Since $det(a) = det(b) = 1$, this preserves the determinant of X, which is a nondegenerate bilinear form on $M_2(C)$. Hence we have a homomorphism $SL(2,C) \times SL(2,C) \to SO(4,C)$. The groups have the same dimension, and the kernel of Lf is empty, and $SO(4,C)$ is connected, so this is a covering map. The kernel is $\{(1,1),(-1,-1)\}$, so it's a double cover.

(b) Let $SL(2,C)$ act on the hermitian 2×2 matrices (which are isomorphic to R^4, not to R^3) by $X \to aXa^*$. As before, this preserves the determinant, which is a bilinear form. This bilinear form has signature (3,1), and the rest of the argument is as in (a).