Lie Groups Solutions, Problem Set # 5

Section 3.1:

1: (a) Given a real vector space E, let $F = E \otimes \mathbf{C}$, and let C be complex conjugation. Conversely, given a pair (F, C) with F of complex dimension n, we can view F as a real vector space of real dimension $2n$. Since $C^2 = 1$, the eigenspaces of C have eigenvalues ± 1, and since $Ci = -iC$, multiplication by i sends each eigenspace to the other. Let E be the $+1$ eigenspace, so iE is the -1 eigenspace, so $F = E \oplus iE = E \otimes \mathbf{C}$. This shows that the correspondence is a bijection.

(b) If E is a right-$\mathbf{H}$ vector space, then E is also a (right) complex vector space (of twice the dimension), since the complexes are a subset of the quaternions. Let F be the same set as E, and let J be right-multiplication by j. Since for any complex number α, $\alpha j = j\bar{\alpha}$, J is complex anti-linear, and since $j^2 = -1$, $J^2 = -1$. Conversely, if we have a pair (F, J), then we can allow the quaternion $q = \alpha + j\beta$ to act on a vector x by $x(\alpha + j\beta) = x(\alpha) + (Jx)\beta$.

Note an essential difference between the two constructions. In the real case, E is a subset of F. In the quaternionic case, E is the same set as F.

6: Here are two solutions: (a) Work on the Lie algebra level. $sl(2, \mathbf{R})$ is spanned by the Hermitian matrices $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and the anti-Hermitian matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Meanwhile, $su(1, 1)$ is spanned by two Hermitian matrices and the anti-Hermitian matrix $B = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. But $A = cBc^{-1}$, where $c = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$ is the matrix of eigenvectors of A. Likewise, the Hermitian generators of $sl(2, \mathbf{R})$ are $Ad(c)$ of the Hermitian generators of $SU(1, 1)$, and by exponentiation we see that $SL(2, \mathbf{R}) = cSU(1, 1)c^{-1}$. (b) Following the hint in the book, $SL(2, \mathbf{R})$, acting on $\mathbf{R}^2$, preserves the bilinear form with matrix $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. But then acting on $\mathbf{C}^2$ it preserves the sesquilinear form with matrix $iM = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. But this is a Hermitian form of signature (1,1), so $SL(2, \mathbf{R}) \subset SU(iM)$. Since the groups are connected and of the same dimension, they are in fact equal. But $SU(iM)$ is conjugate to the standard $SU(1, 1)$ by a change of basis that takes iM to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which is precisely the matrix c listed above.

11: In all cases we count degrees of freedom in the Lie algebra. (a) There are n^2 variables and one constraing (trace equals zero), hence $n^2 - 1$ degrees of freedom. (b)

$so(n, \mathbf{C})$ is the set of anti-symmetric matrices, which are determined by the upper triangular block, with $n(n-1)/2$ degrees of freedom. (c) As we worked out in problem 2.2.3, the Lie algebra of $sp(n, \mathbf{C})$ is all block matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $D = -A^t$ (m^2 degrees of freedom) and with B and C symmetric ($m(m+1)/2$ degrees of freedom each), for a total of $2m^2 + m$.

12: (a) $\mathbf{g}_0 = su(n)$ is the set of traceless anti-Hermitian matrices, $i\mathbf{g}_0$ is the set of traceless Hermitians, and $\mathbf{g}_0 \oplus i\mathbf{g}_0 = \mathbf{g}$ is the set of all traceless matrices. (b) $\mathbf{g}_0$ is the set of anti-symmetric real matrices, $i\mathbf{g}_0$ is the set of anti-symmetric imaginary matrices, and $\mathbf{g} = \mathbf{g}_0 \oplus i\mathbf{g}_0$ is the set of all anti-symmetric matrices. (c) As in problem 2.2.3, $\mathbf{g}_0$ is the set of all real block matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, with $D = -A^t$ and with B and C symmetric, $i\mathbf{g}_0$ is the set of imaginary matrices of that form, and $\mathbf{g}$ is the set of all complex matrices of that form.

Section 3.2:

6: (a) For $G = SL(n, \mathbf{C})$, every simple matrix is diagonalizable, and the matrix of eigenvectors can be chosen to have determinant 1 (just rescale one of the eigenvectors). For the other classical groups, we have a bit more work to do. Suppose that G preserves the bilinear form ϕ. Then for any $a \in G$, and any eigenvectors v_1, v_2 of a with eigenvalues λ_1 and λ_2, $\phi(v_1, v_2) = \phi(av_1, av_2) = \lambda_1\lambda_2\phi(v_1, v_2)$. That is, either $\lambda_1\lambda_2 = 1$ or the two eigenvectors are ϕ-orthogonal. However, ϕ is non-degenerate, so it can't be that EVERY eigenvector is orthogonal to v_j. For each j, there must be an eigenvector w_j whose eigenvalue is λ_j^{-1}, and for which $\phi(v_j, w_j) = 1$. For the moment, suppose all of the eigenvalues of a are distinct, and that $G = SO(2n, \mathbf{C})$ or $Sp(n, \mathbf{C})$. Then we can choose list our eigenvectors in the form $v_1, \ldots, v_n, w_1, \ldots, w_n$. The matrix that has these vectors as its columns will be in G. (Seeing that it preserves ϕ is precisely the ϕ-orthonormality of the eigenvectors. Seeing that it has determinant $+1$ is subtler.) If $G = SO(2n+1, \mathbf{C})$, then there is an additional eigenvector with eigenvalue 1, which goes last. Finally, if there are repeated eigenvalues, then we must do a Gram-Schmidt-like change-of-basis within each eigenspace to ensure that $\phi(v_j, v_k) = 0 = \phi(w_j, w_k)$ and that $\phi(v_j, w_k) = 1$ if $j = k$ and zero otherwise.

(b) If $X \in \mathbf{g}$ is semi-simple, then $\exp(tX)$ is a semi-simple element of G, so by (a) its eigenvectors can be assembled into an element of G. But for t small enough, all eigenvectors of $\exp(tX)$ are eigenvectors of X, so X is conjugate to an element of $\mathbf{h}$ by G.

(c) Pick an element $X \in \mathbf{a}$ such that X has a maximal number of distinct eigenvalues. By (b), X is conjugate (by G) to a diagonal matrix, and without loss of generality we can group the repeated eigenvalues together. Any matrix $Y \in \mathbf{a}$ commutes with

X, and so must be block-diagonal, with blocks corresponding to the eigenspaces of X. Now I claim that the blocks in Y are all proportional to the identity, for otherwise, by first-order perturbation theory, for small t, $X + tY$ would have more distinct eigenvalues than X, which contradicts the maximality condition. Thus every element of $\mathbf{a}$ is diagonal in this basis, so $\mathbf{a}$ is conjugate to a subalgebra of $\mathbf{h}$.

(d) Every connected abelian subgroup A of G consisting only of semi-simple elements is generated by its Lie algebra $\mathbf{a}$, which is, by (c), conjugate to a subalgebra of $\mathbf{h}$. So $A = \Gamma(\mathbf{a})$ is conjugate to a subgroup of $H = \Gamma(\mathbf{h})$.

(e) By (c), there is $g \in G$ such that $Ad(g)\mathbf{a} \subset \mathbf{h}$. But then $\mathbf{a} \subset Ad(g^{-1})\mathbf{h}$. Since $\mathbf{a}$ is a *maximal* abelian subgroup, $\mathbf{a}$ must equal $Ad(g^{-1})\mathbf{h}$, so $\mathbf{a}$ is conjugate to $\mathbf{h}$.

(f) Since A is connected, $A = \Gamma(L(A))$. Note that $L(A)$ is a maximal abelian subgroup consisting of semi-simple elements, so by (e), $L(A)$ is conjugate to $\mathbf{h}$. But then $A = \Gamma(L(A))$ is conjugate to $H = \Gamma(\mathbf{h})$.

(g) Take $G = SO(E)$, and consider A to be the diagonal matrices with entries ± 1, relative to the basis of problem 1. (This is equivalent to changing the bilinear form to the one represented by the identity matrix). These are the only diagonal matrices in G. The only matrices that commute with all of A are diagonal matrices, hence A is a maximal abelian subgroup consisting of semi-simple elements. However A, being finite, is not conjugate to H.

(h) We already did this in class. The group of $2n \times 2n$ matrices with block form $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ is maximal abelian of dimension n^2, but is not conjugate to H (which has dimension $2n$).