Lie Groups Solutions, Problem Set # 6

The problems that say "verify", namely problems 1, 2, 3, 4, 5 and 9 of section 3.3, did not need to be written up, and are not included in these solutions.

Section 3.2:

9: This is a compact version of problem 6, but the results do NOT directly follow from problem 6, since being conjugate to a diagonal matrix in a COMPLEX group does not *a priori* imply being conjugate in a COMPACT subgroup. (It does imply, but the implication takes work to show, which is the point of this exercise.)

A maximal torus is connected, and so is generated by exp of its Lie algebra. Thus the problem is tantamount to showing that $\mathbf{t}$, the Lie algebra of a maximal torus in a compact classical group K, is conjugate to the Lie algebra of the standard Cartan subgroup. This in turn reduces to showing that $\mathbf{t}$ is conjugate to a sub-algebra of the standard Cartan subgroup's Lie algebra (call if $\mathbf{h}$), as maximality will then imply that $\mathbf{t} = \mathbf{h}$.

But that's the same as simultaneously diagonalizing a basis for $\mathbf{t}$, and arranging the eigenvectors into a matrix that is in K. This is done EXACTLY as in the proof of Theorem 1 of section 3.1 (see the book). For $SU(n)$ and $Sp(n)$, this follows from the fact that unitary matrices have orthogonal eigenvectors, and that each pair of eigenvectors of $Sp(n)$ are related by multiplication by j. For $SO(n)$, we have to split each eigenvector into real and imaginary parts.

Section 3.3:

6: We do this first for A_n. Reflection along $\lambda_j - \lambda_{j+1}$ is equivalent to interchanging $\lambda_j \leftrightarrow \lambda_{j+1}$. But ALL permutations can be realized as the product of interchanges of adjacent elements, so the entire Weyl group S_n is generated by reflections along the simple roots.

For B_n (or C_n), we also have reflection along λ_n (or $2\lambda_n$), which just flips the sign of one element (namely λ_n). By permuting elements and flipping signs, we obtain all of W.

For D_n, we have all permutations, generated by reflections about the $\lambda_j - \lambda_{j+1}$, and a specific permutation with two sign flips generated by reflection along $\lambda_{n-1} + \lambda_n$. Combining these we get all permutations together with an even number of sign flips, hence all of W.

7: Again, we begin with A_n. Reflection along $\alpha = \lambda_j - \lambda_{j+1}$ sends the positive root $\lambda_k - \lambda_\ell$ $(k < \ell)$ to itself, unless k or ℓ equals j or $j+1$, But replacing k by $k+1$ or replacing ℓ by $\ell - 1$ only changes the positivity of the root if both $k = j$ and $\ell = j+1$, which is to say that $\lambda_k - \lambda_\ell = \alpha$.

For B_n and C_n the same argument applies when $\alpha = \lambda_j - \lambda_{j+1}$, with the added computation that reflection along $\lambda_j - \lambda_{j+1}$ sends $\lambda_k + \lambda_\ell$ to a sum of two λs, hence a positive root and sends $(2)\lambda_k$ to (twice) another λ. We then only need consider refletions about $\alpha = (2)\lambda_n$, which leaves all the positive roots alone except $\lambda_k \pm \lambda_n$, which goes to $\lambda_k \mp \lambda_n$, and α, which goes to minus itself.

For D_n the argument concerning reflections about $\lambda_j - \lambda_{j+1}$ is the same as before, and we need only check reflections about $\alpha = \lambda_{n-1} + \lambda_n$, which manifestly leaves all positive roots alone except $\lambda_k \pm \lambda_{n-1}$ which goes to $\lambda_k \pm \lambda_n$ (and vice-versa), $\lambda_{n-1} - \lambda_n$, which goes to itself, and α, which flips sign.

(b) Reflection about α sends ρ to $\rho - \alpha$, since it merely permutes the terms in the sum, except for the term $\alpha/2$ which becomes $-\alpha/2$. But by definition reflection about α takes ρ to $\rho - 2[(\rho, \alpha)/(\alpha, \alpha)]\alpha$, so $2[(\rho, \alpha)/(\alpha, \alpha)]\alpha$ must equal 1.

8: (a) A dominant weight pairs non-negatively with every positive root, which is equivalent to pairing non-negatively with each simple root α_j, which is equivalent to being a non-negative linear combination of the dual basis elements α^j.

(b) If λ is higher than 0, it is a non-negative linear combination of positive roots, each of which is a sum of simple roots, so λ is a non-negative linear combination of simple roots. Conversely, simple roots are positive, so every non-negative linear combination of simple roots is higher than zero.