M427K Final Exam Solutions, May 8, 2008

1. a) Consider the (scalar) first-order differential equation $\frac{dy}{dx} = -\frac{3x^2+e^y}{2y+xe^y}$, restricted to the first quadrant ($x \geq 0$, $y \geq 0$). If $y(2) = 0$, what is $y(0)$? [Hint: rewrite the differential equation in exact form]

$(2y + xe^y)dy + (3x^2 + e^y)dx = 0$ is exact, and is equivalent to $d(y^2 + xe^y + x^3) = 0$, or $y^2 + xe^y + x^3 = c$. Plugging in $x = 2, y = 0$ gives $c = 10$, and plugging in $x = 0$ gives $y = \sqrt{10}$. Note that $y(0)$ is NOT equal to $-\sqrt{10}$. Since dy/dx is negative, $y(0) > y(2)$.

b) Consider the differential equation $\frac{dx}{dt} = x + t$ with $x(0) = 0$. Find $x(t)$ for all t. [There are several ways to do this. Any correct method will get full credit.]

This can be solved with integrating factors of by undetermined coefficients. Using integrating factors, we have $x(t) = e^t \int_0^t se^{-s} ds = e^t - t - 1$. Using undetermined coeffients, we guess $y = A + Bt$ and get $A = B = -1$ as a particular solution. So the general solution is $y = Ce^t - t - 1$. Plugging in $y(0) = 0$ gives $C = 1$.

2a. Find the general solution to $y'' - 3y' + 2y = 0$.

$r^2 - 3r + 2 = 0$, so $r = 1$ or $r = 2$, and the general solution is $y = c_1e^t + c_2e^{2t}$.

b) Find a particular solution to $y'' - 3y' + 2y = e^t + e^{3t}$.

We can do this separately for e^t and e^{3t} or all at once. Since $r = 1$ is a root and $r = 3$ isn't, we guess $y = Ate^t + Be^{3t}$. Plug that in and solve for A and B to get $y = -te^t + \frac{1}{2}e^{3t}$ (plus arbitrary multiples of e^t and e^{2t}.

c) Find the general solution to $y'' - 2y' + 2y = 0$.

Now $r = 1 \pm i$, so the general solution is $y = e^t[c_1 \cos(t) + c_2 \sin(t)]$. This can also be expressed as $\tilde{c}_1 e^{(1+i)t} + \tilde{c}_2 e^{(1-i)t}$.

3. **Using the methods of chapter 5**, find a series solution $y = \sum_n a_n x^n$ to $y'' - 3y' + 2y = 0$. More precisely,

a) Find a recursion relation expressing a_n in terms of $a_0, \ldots, a_{n-1}$. If $y(0) = 2$ and $y'(0) = 3$, find $y(0.1)$ to 3 decimal places. [No, you don't need a calculator for this.]

$y'' = \sum(n+2)(n+1)a_{n+2}x^n$, $y' = \sum(n+1)a_{n+1}x^n$ and $y = \sum a_n x^n$. Setting the coefficient of x^{n-2} in $y'' - 3y' + 2y$ equal to zero gives $n(n-1)a_n - 3(n-1)a_{n-1} + 2a_{n-2} = 0$, so $a_n = \frac{3(n-1)a_{n-1} - 2a_{n-2}}{n(n-1)}$. Plugging in $a_0 = 2$ and $a_1 = 3$ then gives $a_2 = 5/2$ and $a_3 = 3/2$, so $y(0.1) \approx a_0 + a_1/10 + a_2/100 + a_3/1000 =$

2.3265. To 3 decimal places, that's either 2.326 or 2.327. Looking at the a_4x^4 term shows that 2.327 is actually closer, but I'll accept either answer.

By the way, the exact solution is $y = e^x + e^{2x}$, whose Taylor series is $2 + 3x + (5/2)x^2 + (3/2)x^3 + \cdots$.

b) Now consider the equation $x^2y'' - 2xy' + (2 + x)y = 0$. For what values of r might a series solution $y = x^r \sum a_n x^n$ (with a_0 nonzero) exist? For the larger value of r, take $a_0 = 1$ and find a_1 and a_2.

The equation for r is $r^2-3r+2 = 0$, so $r = 1$ or 2. If $y = x^r \sum a_n x^n$, setting the coefficient of x^n equal to zero gives $[(n+r)(n+r-3)+2]a_n + a_{n-1} = 0$, or $a_n = -a_{n-1}/[(n + r)(n + r - 3) + 2]$. For $r = 2$ and $a_0 = 1$, $a_1 = -1/2$, $a_2 = (1/2)/6 = 1/12$.

4. a) Find the general solution to the system of ODEs $\frac{dx_1}{dt} = 2x_1 - 2x_2$, $\frac{dx_2}{dt} = x_1 - x_2$. Then find a solution with the initial conditions $x_1(0) = 8$, $x_2(0) = 5$.

The matrix $\begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix}$ has eigenvalues 0 and 1 with eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, so our general solution is $c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t$. This initial conditions say that $c_1 = 2$ and $c_2 = 3$, so $\vec{x} = \begin{pmatrix} 2 + 6e^t \\ 2^3 e^t \end{pmatrix}$.

b) Find the general solution to the system of ODEs $\frac{dx_1}{dt} = 2x_1 - x_2$, $\frac{dx_2}{dt} = 4x_1 - 2x_2$.

The matrix $\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$ has a single eigenvalue (0) with algebraic multiplicity 2 and geometric multiplicity 1. $\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector and $\vec{w} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ is a vector with $A\vec{w} = \vec{v}$. Our general solution is $\vec{x} = c_1\vec{v} + c_2[\vec{w} + t\vec{v}]$.

5. This problem explores how a rectifier (e.g., the AC adapter on your laptop) turns AC current into DC current. The rectifier receives a signal, takes its absolute value, and then passes it through a filter to remove high-frequency components. What's left is close to the constant voltage that your laptop wants. Let $f(x) = \sin(x)$ (that's the wall voltage), and let $g(x) = |f(x)|$. Think of both of them as periodic functions with period 2π.

a) Compute the Fourier coefficients $\hat{f}_n$ for all n.

Since $f(x) = \sin(x) = [e^{ix} - e^{-ix}]/2i$, we have $\hat{f}_{-1} = i/2$, $\hat{f}_1 = -i/2$, and all other Fourier coefficients are zero.

b) Compute the Fourier coefficients $\hat{g}_n$ for all n. [Many of these are zero by symmetry. The rest require integration.]

Note that $g(x) = \sin(x)$ for $0 < x < \pi$ and $g(x) = -\sin(x)$ for $\pi < x < 2\pi$, so $\hat{g}_n = \frac{1}{2\pi}\left[\int_0^\pi e^{-inx}\sin(x)dx - \int_\pi^{2\pi} e^{-inx}\sin(x)dx\right]$. If n is odd, the second integral cancels the first and $\hat{g}_n = 0$. If n is even, they given the same contribution, and $\hat{g}_n = \frac{1}{\pi}\int_0^\pi e^{-inx}\sin(x)dx$. Using the fact that $\sin(x) = [e^{ix} - e^{-ix}]/2i$, this evaluates to $\hat{g}_n = -\frac{2}{\pi(n^2-1)}$.