M427K Final Exam, May 8, 2008

1. a) Consider the (scalar) first-order differential equation $\frac{dy}{dx} = -\frac{3x^2+e^y}{2y+xe^y}$, restricted to the first quadrant ($x \geq 0$, $y \geq 0$). If $y(2) = 0$, what is $y(0)$? [Hint: rewrite the differential equation in exact form]

b) Consider the differential equation $\frac{dx}{dt} = x + t$ with $x(0) = 0$. Find $x(t)$ for all t. [There are several ways to do this. Any correct method will get full credit.]

2a. Find the general solution to $y'' - 3y' + 2y = 0$.

b) Find a particular solution to $y'' - 3y' + 2y = e^t + e^{3t}$.

c) Find the general solution to $y'' - 2y' + 2y = 0$.

3. **Using the methods of chapter 5**, find a series solution $y = \sum_n a_n x^n$ to $y'' - 3y' + 2y = 0$. More precisely,

a) Find a recursion relation expressing a_n in terms of $a_0, \ldots, a_{n-1}$. If $y(0) = 2$ and $y'(0) = 3$, find $y(0.1)$ to 3 decimal places. [No, you don't need a calculator for this.]

b) Now consider the equation $x^2 y'' - 2xy' + (2 + x)y = 0$. For what values of r might a series solution $y = x^r \sum a_n x^n$ (with a_0 nonzero) exist? For the larger value of r, take $a_0 = 1$ and find a_1 and a_2.

4. a) Find the general solution to the system of ODEs $\frac{dx_1}{dt} = 2x_1 - 2x_2$, $\frac{dx_2}{dt} = x_1 - x_2$. Then find a solution with the initial conditions $x_1(0) = 8$, $x_2(0) = 5$.

b) Find the general solution to the system of ODEs $\frac{dx_1}{dt} = 2x_1 - x_2$, $\frac{dx_2}{dt} = 4x_1 - 2x_2$.

5. This problem explores how a rectifier (e.g., the AC adapter on your laptop) turns AC current into DC current. The rectifier receives a signal, takes its absolute value, and then passes it through a filter to remove high-frequency components. What's left is close to the constant voltage that your laptop wants. Let $f(x) = \sin(x)$ (that's the wall voltage), and let $g(x) = |f(x)|$. Think of both of them as periodic functions with period 2π.

a) Compute the Fourier coefficients $\hat{f}_n$ for all n.

b) Compute the Fourier coefficients $\hat{g}_n$ for all n. [Many of these are zero by symmetry. The rest require integration.]