M427K Second Midterm Exam, March 7, 2008

1. Consider the differential equation $y'' - 2y' + 5y = 0$.

a) Find the general solution.

Since the roots of the polynomial $r^2 - 2r + 5 = 0$ are $r = 1 \pm 2i$, our solution is $y(t) = c_1 e^t \cos(2t) + c_2 e^t \sin(2t)$.

b) Find the solution with the initial conditions $y(0) = 1$ and $y'(0) = 5$.

Note that $y(0) = c_1$ and $y'(0) = c_1 + 2c_2$, so $c_1 = 1$ and $c_2 = 2$ and $y(t) = e^t \cos(2t) + 2e^t \sin(2t)$.

c) Find a particular solution to $y'' - 2y' + 5y = 17\cos(2t)$.

Plugging in $y = A\cos(2t) + B\sin(2t)$ gives $y'' - 2y' + 5y = (A - 4B)\cos(2t) + (B + 4A)\sin(2t)$, so $A = 1$ and $B = -4$, and $y = \cos(2t) - 4\sin(2t)$ (plus any answer from part (a)).

2. Consider the differential equation $y''' - 3y'' + 2y' = 0$.

a) Find the general solution.

The roots of $r^3 - 3r^2 + 2r$ are 0, 1, and 2, so our general solution is $y(t) = c_1 + c_2 e^t + c_3 e^{2t}$.

b) Find a particular solution to $y''' - 3y'' + 2y' = 6e^{-t}$.

Try $y = Ae^{-t}$. Then $y''' - 3y'' + 2y' = -6Ae^{-t}$, so $A = -1$ and $y = -e^{-t}$.

c) Find the solution to $y''' - 3y'' + 2y' = 6e^{-t}$ with initial conditions $y(0) = 18$, $y'(0) = 25$, $y''(0) = 35$.

We have $y(t) = -e^{-t} + c_1 + c_2 e^t + c_3 e^{2t}$, so $y(0) = -1 + c_1 + c_2 + c_3$, $y'(0) = 1 + c_2 + 2c_3$ and $y''(0) = -1 + c_2 + 4c_3$. Setting these equal to 18, 25, and 35 gives $c_1 = 1$, $c_2 = 12$ and $c_3 = 6$, so $y(t) = -e^{-t} + 1 + 12e^t + 6e^{2t}$.

3a. Let A be a 4 by 6 matrix whose reduced row-echelon form is

$$A_{RREF} = \begin{pmatrix} 1 & 2 & 0 & 3 & 0 & 5 \\ 0 & 0 & 1 & 4 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Find all solutions to $A\vec{x} = 0$.

Read off the equations from the rows of A_{RREF}:

$$\begin{aligned} x_1 &= -2x_2 - 3x_4 - 5x_6 \\ x_2 &= x_2 + 0x_4 + 0x_6 \\ x_3 &= 0 - 4x_4 - 6x_6 \\ x_4 &= 0 + x_4 + 0 \\ x_5 &= 0 + 0 - 7x_6 \\ x_6 &= 0 + 0 + x_6 \end{aligned}$$

That is $\vec{x} = c_1(-2, 1, 0, 0, 0, 0)^T + c_2(-3, 0, -4, 1, 0, 0)^T + c_3(-5, 0, -6, 0, -7, 1)^T$.

b) Consider the matrix $A = \begin{pmatrix} 1 & 1 & 1 & -2 \\ 1 & 2 & 4 & -3 \\ 2 & 4 & 9 & -7 \\ 4 & 7 & 14 & -12 \end{pmatrix}$. Are the columns of this matrix linearly independent? If so, express $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ as a linear combination of the columns of A. If not, find a nontrivial linear combination of the columns that equals zero.

Row-reducing A gives $\begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, so the columns are linearly dependent. Specifically $3\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3 + \vec{v}_4 = 0$, where $\vec{v}_i$ is the i-th column of A.

4a) Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 4 & 2 \\ 5 & 1 \end{pmatrix}$.

The characteristic polynomial is $\begin{vmatrix} \lambda - 4 & -2 \\ -5 & \lambda - 1 \end{vmatrix} = \lambda^2 - 5\lambda - 6$, whose roots are 6 and -1. The corresponding eigenvectors are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$.

b) The matrix $\begin{pmatrix} 6 & 5 \\ 2 & 3 \end{pmatrix}$ has eigenvalues 1 and 8, with corresponding eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$. Find the general solution to the equations

$$\begin{aligned} x_1' &= 6x_1 + 5x_2 \\ x_2' &= 2x_1 + 3x_2. \end{aligned}$$

$$\vec{x} = c_1 e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{8t} \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$

c) Find the solution with initial conditions $x_1(0) = 7$, $x_2(0) = 7$.

Since $\vec{x}(0) = \begin{pmatrix} c_1 + 5c_2 \\ -c_1 + 2c_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$, we must row-reduce $\left(\begin{array}{cc|c} 1 & 5 & 7 \\ -1 & 2 & 7 \end{array}\right)$ to get $c_1 = -3$ and $c_2 = 2$, so $\vec{x}(t) = -3e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2e^{8t} \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, or equivalently $x_1 = -3e^t + 10e^{8t}$ and $x_2 = 3e^t + 4e^{8t}$.