M427K Third Midterm Exam Solutions, April 30, 2008

1. Consider the nonlinear system of differential equations

$$\frac{dx_1}{dt} = x_1(1 - x_1 - 2x_2)$$
$$\frac{dx_2}{dt} = -x_2(1 - \frac{x_1}{2})$$

a) Find the fixed points.

Either $x_1 = 0$ or $(1 - x_1 - 2x_2) = 0$, and either $x_2 = 0$ or $1 - x_1/2 = 0$. Since you can't have x_1 and $1 - x_1/2$ both equaling zero, there are three possibilities, namely (0,0), (1,0) and $(2, -\frac{1}{2})$.

b) For each fixed point, find a linear system of differential equations that approximates the system near the fixed point.

For each fixed point $\vec{a}$, let $\vec{y} = \vec{x} - \vec{a}$, and we have the approximate equations $\frac{d\vec{y}}{dt} = A\vec{y}$, where $A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}$, evaluated at $\vec{a}$. This gives $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & -2 \\ 0 & -\frac{1}{2} \end{pmatrix}$, and $\begin{pmatrix} -2 & -4 \\ -\frac{1}{4} & 0 \end{pmatrix}$ at the three fixed points, respectively.

c) For each fixed point, indicate whether the point is a source, sink, saddle point, spiral (in or out?), or is borderline.

The eigenvalues of the first matrix are ± 1, so (0,0) is a saddle. The eigenvalues of the second are -1 and $-\frac{1}{2}$, so (1,0) is a sink. The eigenvalues of the third are $-1 \pm \sqrt{2}$, so $(2, -\frac{1}{2})$ is a saddle.

2. Consider the differential equation $y'' + \sin(x)y' + \cos(x)y = 0$. Recall that $\sin(x) = x + O(x^3)$ and $\cos(x) = 1 - \frac{x^2}{2} + O(x^4)$. We seek solutions of the form $y = \sum_{n=0}^{\infty} a_n x^n$.

a) If $y(0) = 1$ and $y'(0) = 0$, find a_0, a_1, a_2, a_3 and a_4.

$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + O(x^5)$, so $y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + O(x^4)$ and $y''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + O(x^3)$. Since $\sin(x) = x + O(x^3)$, $\sin(x)y' = a_1 x + 2a_2 x^2 + O(x^3)$. Since $\cos(x) = 1 - \frac{1}{2}x^2 + O(x^4)$, $\cos(x)y = a_0 + a_1 x + (a_2 - \frac{a_0}{2})x^2 + O(x^3)$. The entire right hand side of the differential equation can be written as

$y'' + \sin(x)y' + \cos(x)y = 2a_2 + a_0 + (2a_1 + 6a_3)x + (12a_4 + 3a_2 - \frac{a_0}{2})x^2 + O(x^3)$. This implies that $a_2 = -a_0/2$, $a_3 = -a_1/3$ and $a_4 = (a_0 - 6a_2)/24 = a_0/6$.

If $y(0) = 1$ and $y'(0) = 0$, then $a_0 = 1$, $a_1 = 0$, $a_2 = -\frac{1}{2}$, $a_3 = 0$, and $a_4 = \frac{1}{6}$. In other words, $y = 1 - \frac{x^2}{2} + \frac{x^4}{6} + O(x^5)$.

b) If $y(0) = 0$ and $y'(0) = 1$, find a_0, a_1,a_2, a_3 and a_4.

If $y(0) = 0$ and $y'(0) = 1$, then $a_0 = 0$, $a_1 = 1$, $a_2 = 0$, $a_3 = -1/3$, and $a_4 = 0$. In other words, $y = x - \frac{x^3}{3} + O(x^5)$.

3. Now consider the differential equation $x^2y'' + xy' + (x^2 - 2)y = 0$. (This is a special case of Bessel's equation.) For $x > 0$, we seek solutions of the form $y = x^r(a_1 + a_1x + a_2x^2 + \cdots)$, with a_0 nonzero.

a) For what values of r do such solutions exist?

Our equation for r is $r^2 - 2 = 0$, so $r = \pm\sqrt{2}$.

b) For the largest value of r, find a recursion relation expressing a_n in terms of $a_0, a_1, \ldots, a_{n-1}$.

If $y = x^r \sum a_n x^n = \sum_n a_n x^{n+r}$, then $xy' = x^r \sum (n+r)a_n x^n$, $x^2y'' = x^r \sum (n+r)(n+r-a)a_n x^n$, and $(x^2 - 2)y = x^r \sum (a_{n-2} - 2a_n)x^n$ (with $a_{-2} = a_{-1} = 0$).

Plugging this into the equation gives $a_n[(n+r)^2 - 2] + a_{n-2} = 0$. For $n = 0$ this implies that $r = \pm\sqrt{2}$. For $n = 1$ it implies that $a_1 = 0$, and hence that all the odd a_n vanish. For higher n we have $a_n = \frac{-a_{n-2}}{(n+r)^2-2}$.

b) For the largest value of r, set $a_0 = 1$ and find a_1, a_2 and a_3.

$a_1 = a_3 = 0$ and $a_2 = -\frac{1}{(2+\sqrt{2})^2-2} = -\frac{1}{2} + \frac{\sqrt{2}}{4}$.

4. On the interval $[0, 1]$, we seek to expand the function

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 0 & \text{if } x = 1 \end{cases}$$

as a Fourier sine series $f(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)$.

a) Find c_n for all n. [You may find the identity $\int x \sin(ax)dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a}$ to be useful]

$c_n = 2\int_0^1 x \sin(n\pi x)dx = -\frac{2\cos(n\pi)}{n\pi}$. This equals $2/n\pi$ if n is odd and $-2/n\pi$ if n is even.

b) Evaluate this series at $x = 1/2$ to obtain a formula for π as an infinite sum of rational numbers.

$\frac{1}{2} = f(\frac{1}{2}) = \sum c_n \sin(n\pi/2) = c_1 - c_3 + c_5 - c_7 + \cdots = \frac{2}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$, so $\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$.