M427K First Midterm Exam, February 6, 2008

1. Consider the differential equation

$$\frac{dy}{dx} = -\frac{x}{y}, \qquad y(-1) = 3.$$

Find $y(3)$.

2. There is a radioactive material A that breaks down to material B at a rate r. That is, if $x(t)$ represents the amount of material A at time t, then $dx/dt = -rx$. B breaks down to C, also at rate r. We start with 1 kg of A, no B, and no C.

a) Find $x(t)$ for all time.

b) Let $y(t)$ be the amount of material B at time t. Write down a differential equation for y, relating dy/dt to x and y.

c) Plug in your answer for part (a) to part (b) to get a differential equation for y, relating dy/dt to y and t. The variable x should no longer appear in this equation.

d) Solve this equation to get $y(t)$ as an explicit function of t. (Note that $y(0) = 0$.)

3. Consider the differential equation $\frac{dy}{dx} = 2xy$ with initial condition $y(0) = 1$.

a) Rewrite this as an integral equation.

b) Find three approximate solutions to this equation by Picard iteration, starting with $y_0(t) = 1$. That is, write down $y_1(t)$, $y_2(t)$ and y_3. (Don't worry about what interval you're working on – this procedure converges for all values of t.)

c) Solve the differential equation *exactly*, using whatever method you wish. The first few terms in the Taylor series for this exact solution should agree with the approximate solutions you found in (b).

4a) Consider the differential equation $\frac{dx}{dt} = \frac{1}{2}\sin\left(\frac{x^2}{\pi}\right)$. Find all the fixed points and indicate which are stable and which are unstable. (Warning: some fixed points may be neutral.)

b) Now consider the *difference* equation $x_{n+1} = x_n + \frac{1}{2}\sin\left(\frac{x_n^2}{\pi}\right)$. Find all the fixed points and indicate which are stable and which are unstable. (Same warning as before.) You do *not* have to find and classify the periodic points; just the fixed points.