

Differential Topology

Operations on vector bundles, and
Homework 11, due May 8

Let X be a smooth manifold of dimension n , and suppose that E_1 and E_2 are vector bundles over X with dimension (also called rank) k_1 and k_2 . (They can be real vector bundles or complex vector bundles – the following constructions work equally well in both cases, although we do require both bundles to live over the same field.) If $\{U_\alpha\}$ is an open cover of X , then we have trivialisations for both bundles over each set U_α , and transition functions $g_{\beta\alpha}^{(1)}$ and $g_{\beta\alpha}^{(2)}$ for each nontrivial intersection $U_\alpha \cap U_\beta$.

The **direct sum** $E_1 \oplus E_2$ is a bundle of rank $k_1 + k_2$, whose transition functions are $\begin{pmatrix} g_{\beta\alpha}^{(1)} & 0 \\ 0 & g_{\beta\alpha}^{(2)} \end{pmatrix}$. The **tensor product** $E_1 \otimes E_2$ is a bundle of rank $k_1 k_2$ with transition functions $g_{\beta\alpha}^{(1)} \otimes g_{\beta\alpha}^{(2)}$. That is, $g_{\beta\alpha}(\mathbf{v} \otimes \mathbf{w}) = g_{\beta\alpha}^{(1)}(\mathbf{v}) \otimes g_{\beta\alpha}^{(2)}(\mathbf{w})$.

We also have a bundle of homomorphisms $Hom(E_1, E_2)$, whose fiber over x is the set of linear transformations from the fiber of E_1 over x to the fiber of E_2 over x . This is generally a different rank $k_1 k_2$ vector bundle than $E_1 \otimes E_2$.

Problem 1 Describe the transition functions for $Hom(E_1, E_2)$ in terms of the transition functions of E_1 and those of E_2 .

An important special case is where E_2 is the trivial rank-1 vector bundle $X \times \mathbb{R}$ or $X \times \mathbb{C}$. In that case, the fiber of $Hom(E_1, E_2)$ over x is just the dual space to the fiber of E_1 . We denote this dual bundle E_1^* .

We say that two bundles $E_{1,2}$ over X are isomorphic if there is a section of $Hom(E_1, E_2)$ that is everywhere invertible.

Problem 2 Describe the transition functions of E_1^* in terms of the transition functions of E_1 . Show that $Hom(E_1, E_2)$ is isomorphic to $E_2 \otimes E_1^*$.

A rank-1 vector bundle is called a **line bundle**. If L is a line bundle, then so is $L \otimes L$, which we denote L^2 , and similarly $L^{n+1} = L \otimes L^n$. We write L^{-1} for L^* , and define a negative power of L to be a positive power of L^{-1} . Also, if E is any vector bundle, then the **determinant line bundle** $det(E)$ is a line bundle whose transition functions are $det(g_{\beta\alpha})$.

We next consider connections on vector bundles.

Problem 3. Suppose that ∇_0 and ∇_1 are two connections on the same

bundle E over X . Show that $\nabla_t = t\nabla_1 + (1-t)\nabla_0$ is also a connection. This shows that the set of connections on a bundle is connected.

We showed in class that, if X is an oriented 2-manifold, and if L is a complex line bundle, then $\int_X F$ is a multiple of $2\pi i$. Since this multiple varies continuously with the connection, yet is always an integer, it must be the same for all connections on a line bundle. In other words, it's the sort of topological invariant, like Euler characteristic or degree, that we've been studying all term. This multiple is called the (first) **Chern number** of the line bundle, and we use it to distinguish line bundles. For a higher rank complex vector bundle E , we define the first Chern number of E to be the first Chern number of $\det(E)$. (The curvature of this determinant bundle is the trace of the curvature of our connection on E , so the first Chern number is obtained by integrating the trace of the curvature on E and dividing by $2\pi i$.)

Problem 4 Show that, if L and L' are line bundles over X with connections with curvatures F and F' , and if $\int_X F \neq \int_X F'$, then L is not isomorphic to L' . [Hint: Show that, given an isomorphism $L \rightarrow L'$, you can pull back a connection on L' to a connection on L whose curvature is related to the curvature of L' .]

Our next task is to classify complex vector bundles over S^2 , up to isomorphism. We begin with line bundles, starting with an example. Consider $\mathbb{C}P^1 = S^2$, with two coordinate charts $U_1 = \{[1, z]\}$, where z ranges over the complex numbers, and $U_2 = \{[w, 1]\}$ where w ranges over the complex numbers. Of course, $w = z^{-1}$ on the overlap of these two sets. (The bracket means the equivalence class, so $[z_1, z_2] = [\lambda z_1, \lambda z_2]$ for any nonzero complex number λ .)

Over $\mathbb{C}P^1$ there is a tautological line bundle L_1 , a sub-bundle of $\mathbb{C}P^1 \times \mathbb{C}^2$, whose fiber over $[z_1, z_2]$ is all multiples of the vector (z_1, z_2) . Let $L_n = L_1^n$.

Problem 5. Find local trivializations for L_n over U_1 and U_2 , and give the corresponding transition functions.

Problem 6. Define a connection on L_1 as a sub-bundle of the trivial \mathbb{C}^2 bundle. To take a derivative of a section of L_1 , take its derivative as a section of $\mathbb{C}P^1 \times \mathbb{C}^2$. Express this connection in terms of the trivializations you found in problem 5, compute the curvature, and integrate this curvature over $\mathbb{C}P^1$.

Problem 7. Given a connection on bundles E_1 and E_2 , we can define a connection on $E_1 \otimes E_2$ by $\nabla(s_1 \otimes s_2) = (\nabla s_1) \otimes s_2 + s_1 \otimes \nabla s_2$. If E_1 and E_2 are line bundles, how does the curvature of this connection on $E_1 \otimes E_2$ relate to the curvatures of the connections on E_1 and E_2 ? Based on this calculation, what is the Chern number of L_n ?

This shows that the bundles L_n are all distinct. In fact, every line bundle over S^2 is isomorphic to one of these. To see this, note that any line bundle can be viewed as a trivial bundle over the northern hemisphere, a trivial bundle over the southern hemisphere, and a transition function that maps a neighborhood of the equator (i.e., something that has the homotopy type of a circle) to $\mathbb{C} - \{0\}$. This transition function has a winding number, and if that winding number is n then it is homotopic to the winding number of L_{-n} . (Actually, there's a sign ambiguity here, since I didn't say whether you should use the one from the northern patch to the southern patch, or the one from the south to the north. Pick the direction that makes this convention work!) With a little work (which I'm **not** assigning as homework), you can show that every bundle is therefore isomorphic to L_{-n} for some n .

Next we consider higher-rank vector bundles over S^2 , or more generally over any oriented 2-manifold X :

Problem 8 Suppose E is a complex vector bundle over X of rank $k > 1$. Show that there is a nonvanishing section of E . Use this to show that E is isomorphic to $\mathbb{C} \oplus E'$, where E' is a vector bundle of rank $k - 1$.

By peeling off factors of \mathbb{C} until we are left with a line bundle, we obtain a classification of **all** vector bundles over S^2 : Two vector bundles are isomorphic if and only if (1) they have the same rank, and (2) they have the same Chern number.

The same classification works over any oriented 2-manifold. For manifolds of higher dimension, the first Chern **class** is a 2-cohomology class (represented by the trace of F divided by $2\pi i$). There is also a second Chern class represented by a 4-form, a third Chern class represented by a 6-form, and so on. Two complex vector bundles are isomorphic if and only if their ranks are the same and all of their Chern classes are the same.

The classification of real vector bundles is more complicated, and involves cohomology classes of degree 4, 8, 12, etc., as well as something called the Euler class. We're not going to get into them.