Name:

M340L Final Exam Solutions
May 13, 1995

Problem 1: Find all solutions (if any) to the system of equations. Express your answer in vector parametric form.

$$\begin{aligned} x_1 \quad\quad + 2x_3 + 3x_4 &= 6 \\ 2x_1 + 2x_2 + x_3 - 3x_4 &= 2 \\ 4x_1 + 2x_2 + 5x_3 + 3x_4 &= 14 \end{aligned}$$

$$\text{The matrix } \begin{pmatrix} 1 & 0 & 2 & 3 & | & 6 \\ 2 & 2 & 1 & -3 & | & 2 \\ 4 & 2 & 5 & 3 & | & 14 \end{pmatrix} \text{ row-reduces to } \begin{pmatrix} 1 & 0 & 2 & 3 & | & 6 \\ 0 & 1 & -3/2 & -9/2 & | & -5 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix},$$

in other words,

$$\begin{aligned} x_1 &= -2x_3 - 3x_4 + 6 \\ x_2 &= 1.5x_3 + 4.5x_4 - 5 \\ x_3 &= x_3 \\ x_4 &= \quad\quad x_4 \end{aligned},$$

hence $\mathbf{x} = s\begin{pmatrix} -2 \\ 3/2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ 9/2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 6 \\ -5 \\ 0 \\ 0 \end{pmatrix}$, where s and t are arbitrary constants.

Problem 2: T is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$ defined by $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_1 - x_2 \\ x_2 \\ x_1 + x_2 \end{pmatrix}$.

a) Find the matrix of this linear transformation.

Ans: $\begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$, which has rank 3, as can be seen by row reduction.

b) Is T 1-1? If not, find a nonzero vector $\mathbf{x}$ such that $T(\mathbf{x}) = 0$.

Yes, since there is a pivot in each column.

c) Is T onto? If not, find a nonzero vector $\mathbf{y}$ such that $\mathbf{y}$ is not in the range of T.

No, as there are only 3 pivots and 4 rows. The vector $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}$ is not in the range.

Problem 3: a) Compute the determinant of the matrix

$$A = \begin{pmatrix} 4 & 8 & 8 & 8 & 5 \\ 0 & 1 & 0 & 0 & 0 \\ 6 & 8 & 8 & 8 & 7 \\ 0 & 8 & 8 & 3 & 0 \\ 0 & 8 & 2 & 0 & 0 \end{pmatrix}$$

There are many ways to do this. One is to row-reduce A. Another is to write

$$det(A) = \begin{vmatrix} 4 & 8 & 8 & 5 \\ 6 & 8 & 8 & 7 \\ 0 & 8 & 3 & 0 \\ 0 & 2 & 0 & 0 \end{vmatrix} = 2 \begin{vmatrix} 4 & 8 & 5 \\ 6 & 8 & 7 \\ 0 & 3 & 0 \end{vmatrix} = -6 \begin{vmatrix} 4 & 5 \\ 6 & 7 \end{vmatrix} = 12,$$

where we expanded by minors about the second row, then about the last row, then about the last remaining row, and then used a criss-cross on the resulting 2 by 2 determinant.

b) Is A invertible? Why or why not?

Since the determinant is nonzero, A is invertible.

c) What is the rank of A?

Since A is invertible, it must have rank 5.

Problem 4: Let $\mathcal{E} = \{1, t, t^2\}$ be the standard basis for $\mathbb{P}_2$.
Let $\mathcal{B} = \{1 + t + t^2, 1 + 2t + 3t^2, 1 + 4t + 9t^2\}$ be another basis.
Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the linear transformation $T(p(t)) = p(t) + t(dp(t)/dt)$.

a) Find the matrix of T relative to the standard basis $\mathcal{E}$. Call this matrix A.

We compute $T(1) = 1$, $T(t) = 2t$, and $T(t^2) = 2t^2$, so the matrix of T with respect to the standard $\mathcal{E}$ basis is just $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The basis $\mathcal{E}$ is actually a basis of eigenvectors.

b) Find the matrix of T relative to the basis $\mathcal{B}$. Call this matrix B.

Again we compute the action of T on our basis: $T(\mathbf{b}_1) = T(1 + t + t^2) = 1 + 2t + 3t^2 = \mathbf{b}_2$. $T(\mathbf{b}_2) = 1 + 4t + 9t^2 = \mathbf{b}_3$. $T(\mathbf{b}_3) = 1 + 8t + 27t^2 = 6\mathbf{b}_1 - 11\mathbf{b}_2 + 6\mathbf{b}_3$, so

$$B = \begin{pmatrix} [T(\mathbf{b}_1)]_{\mathcal{B}} & [T(\mathbf{b}_2)]_{\mathcal{B}} & [T(\mathbf{b}_3)]_{\mathcal{B}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 6 \\ 1 & 0 & -11 \\ 0 & 1 & 6 \end{pmatrix}.$$

c) Write down the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{E}$. Call this matrix P.

$$P = P_{\mathcal{E}\mathcal{B}} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}.$$

d) Write an equation expressing B in terms of A and P.

$B = P^{-1}AP$. The difference between this and our usual PDP^{-1} formula is that $\mathcal{E}$, and not $\mathcal{B}$, is our basis of eigenvectors, so we are doing a change-of-basis in the opposite direction as usual.

Problem 5: The following matrices are row-equivalent:

$$A = \begin{pmatrix} 1 & 1 & 5 & 1 & 8 \\ 1 & 2 & 7 & 0 & 7 \\ 1 & 3 & 9 & 0 & 10 \\ 2 & 4 & 14 & 1 & 18 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

a) Find a basis for the row space of A.

The nonzero rows of B, namely $(1, 0, 3, 0, 1), (0, 1, 2, 0, 3), (0, 0, 0, 1, 4)$.

b) Find a basis for the column space of A.

The pivot columns of A, namely $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

c) Find a basis for the null space of A.

$$\left\{ \begin{pmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ 0 \\ -4 \\ 1 \end{pmatrix} \right\}.$$

Problem 6: Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

a) Find all the real eigenvalues of A and the corresponding eigenvectors.

The characteristic equation is $(\lambda - 3)(\lambda^2 - 2\lambda + 5) = 0$, whose only real root is 3, with corresponding eigenvector $(0, 1, 0)^T$. b) Find all *complex* eigenvalues and corresponding eigenvectors of A.

The complex eigenvalues are $1 \pm 2i$, with corresponding eigenvectors $(\pm i, 0, 1)^T$. Or, if you prefer the first component to be real, eigenvectors $(1, 0, \mp i)^T$.

c) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. [Warning: P and D may not be real].

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1+2i & 0 \\ 0 & 0 & 1-2i \end{pmatrix}, \text{ and with my choice of eigenvectors, } P = \begin{pmatrix} 0 & i & -i \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Problem 7: In contrast to predator-prey relationships, it can sometimes happen that two different kinds of animals actively help each other. This is called symbiosis. Since I can't think of any realistic examples, I'll illustrate with mythical animals.

Griffins and Dragons live in the Enchanted Forest. Dragons can live without griffins, but griffins cannot live without dragons. The number of dragons D and griffins G in the forest each year is determined by the populations the previous year, according to the formulas:

$$\begin{aligned} D_{k+1} &= 1.5D_k + G_k, \\ G_{k+1} &= D_k \end{aligned}$$

The eigenvalues of the matrix $\begin{pmatrix} 1.5 & 1 \\ 1 & 0 \end{pmatrix}$ are 2 and $-.5$, with corresponding eigenvalues $(2,1)$ and $(1,-2)$.

a) If in year 0 there are 25 dragons and no griffins, what will the populations be in year k? (Don't worry about your answers being fractional. Mythical animals don't have to come in whole units).

Letting $\mathcal{B}$ be the basis of eigenvectors, letting $\mathbf{x}(k) = \begin{pmatrix} D_k \\ G_k \end{pmatrix}$, and $\mathbf{y}(k) = [\mathbf{x}(k)]_{\mathcal{B}}$, we have $P = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$, $P^{-1} = \frac{1}{5}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$, $\mathbf{x}(0) = (25,0)^T$, $\mathbf{y}(0) = P^{-1}\mathbf{x}(0) = (10,5)^T$, $\mathbf{y}(k) = \begin{pmatrix} 10 \cdot 2^k \\ 5 \cdot (-0,5)^k \end{pmatrix}$, and $\mathbf{x}(k) = 10 \cdot 2^k \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 5 \cdot (-0.5)^k \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

b) In the long run, will the populations grow, shrink, or approach a nonzero equilibrium value?

The biggest eigenvalue is larger than 1, so the system will grow, with the component in the $\mathbf{b}_1$ direction doubling every year.

c) After a long time, approximately what will the ratio of dragons to griffins be?

In the long run, $\mathbf{x}(k)$ will point in the $\mathbf{b}_1$ direction, with twice as many dragons as griffins.

Problem 8: Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors $(1,1,1,1)$, $(1,2,2,3)$, and $(5,7,7,9)$.

a) What is the dimension of W?

We can either represent W as the row space of a 3×4 matrix $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 3 \\ 5 & 7 & 7 & 9 \end{pmatrix}$ or as the column space of a 4×3 matrix $\begin{pmatrix} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 1 & 2 & 7 \\ 1 & 3 & 9 \end{pmatrix}$. Either way, row reduction shows that the rank is 2, so W is 2-dimensional. You can also get this result by doing Gram-Schmidt and computing $\mathbf{y}_3 = 0$, indicating that $\mathbf{x}_3$ is in the span of $\mathbf{y}_1$ and $\mathbf{y}_2$, hence in the span of $\mathbf{x}_1$ and $\mathbf{x}_2$.

b) Find an orthonormal basis for W.

We get an orthogonal basis by Gram-Schmidt, with $\mathbf{y}_1 = \mathbf{x}_1 = (1,1,1,1)$ and $\mathbf{y}_2$ being the part of $\mathbf{x}_2 = (1,2,2,3)$ that's orthogonal to $\mathbf{y}_1$, namely $(-1,0,0,1)$. Normalizing, we get an orthonormal basis $\frac{1}{2}(1,1,1,1)$, $\frac{\sqrt{2}}{2}(-1,0,0,1)$.

Problem 9: Let W be the subspace of $\mathbb{R}^4$ spanned by $\mathbf{u}_1 = (1,0,1,0)$ and $\mathbf{u}_2 = (4,3,-4,3)$. (Note that $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal).

a) Find the orthogonal projection of the vector $(3,0,3,5)$ onto the plane W.

Let $\mathbf{x} = (3,0,3,5)$. Since $\mathbf{x} \cdot \mathbf{u}_1 = 6$, $\mathbf{u}_1 \cdot \mathbf{u}_1 = 2$, $\mathbf{x} \cdot \mathbf{u}_2 = 15$ and $\mathbf{u}_2 \cdot \mathbf{u}_2 = 50$, our orthogonal projection is $\hat{\mathbf{x}} = \frac{6}{2}\mathbf{u}_1 + \frac{15}{50}\mathbf{u}_2 = \begin{pmatrix} 4.2 \\ 0.9 \\ 1.8 \\ 0.9 \end{pmatrix}$.

b) Find the distance from W to the point $(3,0,3,5)$.

This is the length of $\mathbf{x} - \hat{\mathbf{x}} = (-1.2, -0.9, 1.2, 4.1)^T$, which works out to be 4.5, via some remarkably grungy arithmetic.

b) Find a least-squares solution to the system of equations

$$\begin{pmatrix} 1 & 4 \\ 0 & 3 \\ 1 & -4 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 3 \\ 5 \end{pmatrix}$$

You can either do this naively by computing $A^T A = \begin{pmatrix} 2 & 0 \\ 0 & 50 \end{pmatrix}$ and $A^T \mathbf{b} = \begin{pmatrix} 6 \\ 15 \end{pmatrix}$ to get the solution $\begin{pmatrix} 3 \\ 3/10 \end{pmatrix}$, or notice that it's really the same question as part (a), namely finding the coefficients when we expand $\hat{\mathbf{x}}$ in the $\{\mathbf{u}_1, \mathbf{u}_2\}$ basis.

Problem 10. True of False

a) A 6×5 matrix cannot have a pivot position is every row.

TRUE. There are only 5 columns, so you can't have 6 pivots.

b) Let v_1, v_2 and v_3 be vectors in $\mathbb{R}^3$. If none of these vectors is a multiple of one of the others, then the set is linearly independent.

FALSE. Consider the vectors $(1,-1,0)^T$, $(-1,0,1)^T$ and $(0,1,-1)^T$, no two of which are parallel, but whose sum is zero.

c) If a system $Ax = b$ has more than one solution, then $Ax = 0$ has more than one solution.

TRUE. If $Ax = b$ has multiple solutions, then there must be free variables, so $Ax = 0$ has multiple solutions.

d) If $AB = BA$ and A is invertible, then $A^{-1}B = BA^{-1}$.

TRUE. $A^{-1}B = A^{-1}BAA^{-1} = A^{-1}ABA^{-1} = BA^{-1}$.

e) If $AB = 0$, then either $A = 0$ or $B = 0$.

FALSE. Consider $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, or even $A = B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

f) For every real matrix A, $\det(A^TA) \geq 0$.

TRUE. $\det(A^TA) = \det(A^T)\det(A) = (\det(A))^2 \geq 0$.

g) Let S be a set of vectors in a vector space V. If $\text{Span}(S) = V$, then a subset of S is a basis for V.

TRUE. This is the spanning set theorem.

h) A change-of-basis matrix is always invertible.

TRUE. You can alwals convert back.

i) If a square matrix A is diagonalizable, then its columns are linearly independent.

FALSE. If a matrix is *invertible* then its columns are linearly independent. Invertibility and diagonalizability have nothing to do with each other. As an example, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is diagonalizable (in fact it's already diagonal), but its columns are linearly dependent.

j) If W is a subspace of $\mathbb{R}^n$, then W and $W^\perp$ have no nonzero vectors in common.

TRUE. If a vector is in both W and $W^\perp$, it has to be perpendicular to itself, which is only possible for the zero vector.