M340L Midterm Exam Solutions
April 11, 1995

Problem 1: Consider the matrices

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & -2 \\ 3 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 3 & 1 \end{pmatrix}$$

a) Compute $\det(A)$ and $\det(B)$.

b) Is A invertible? If so, compute $\det(A^{-1}B)$. If not, find a nonzero vector in $Nul(A)$. Note: There is an easy way to do this part of the problem.

Ans: $\det(A) = 2$, $\det(B) = -7/2$. Since $\det(A) \neq 0$, A is invertible and $\det(A^{-1}B) = \det(A)^{-1}\det(B) = -7/2$.

Problem 2 Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 & 3 \\ 2 & 4 & 3 & 3 & 7 \\ 2 & 4 & 4 & 6 & 8 \end{pmatrix}$$

a) What is the rank of A? b) What is the dimension of the null space of A? The row space? c) Find a basis for the column space of A. d) Find a basis for the null space of A.

Ans: A row-reduces to $\begin{pmatrix} 1 & 2 & 0 & -3 & 2 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. There are two nonzero pivots, so the rank of A is 2. The dimension of the row space is the same as the rank (2). And the dimension of the null space is $5 - 2 = 3$.

The pivots are in the 1st and 3rd columns, so $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$ is a basis for the column space.

A basis for the null space is $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$.

Problem 3. Let $\mathcal{D} = \{d_1, d_2, d_3\}$ and $\mathcal{F} = \{f_1, f_2, f_3\}$ be bases for a 3-dimensional vector space V, and suppose that $f_1 = 2b_1 - b_2 + b_3$, $f_2 = 3b_2 + b_3$, $f_3 = 4b_1 + 2b_3$.

a) Find the change-of-coordinates matrix from $\mathcal{F}$ to $\mathcal{D}$.

$$P_{\mathcal{D} \leftarrow \mathcal{F}} = ([f_1]_{\mathcal{D}}, [f_2]_{\mathcal{D}}, [f_3]_{\mathcal{D}}) = \begin{pmatrix} 2 & 0 & 4 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

b) Find the change-of-coordinates matrix from $\mathcal{D}$ to $\mathcal{F}$.

$$P_{\mathcal{F} \leftarrow \mathcal{D}} = (P_{\mathcal{D} \leftarrow \mathcal{F}})^{-1} = \begin{pmatrix} -3/2 & -1 & 3 \\ -1/2 & 0 & 1 \\ 1 & 1/2 & -3/2 \end{pmatrix}$$

c) If $[x]_{\mathcal{F}} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, find $[x]_{\mathcal{D}}$.

$$[x]_{\mathcal{D}} = P_{\mathcal{D}\leftarrow\mathcal{F}}[x]_{\mathcal{F}} = \begin{pmatrix} 14 \\ 5 \\ 9 \end{pmatrix}$$

d) If $[y]_{\mathcal{D}} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}$, find $[y]_{\mathcal{F}}$.

$$[y]_{\mathcal{F}} = P_{\mathcal{F}\leftarrow\mathcal{D}}[y]_{\mathcal{D}} = \begin{pmatrix} 9 \\ 3 \\ -5 \end{pmatrix}$$

Problem 4. Let $A = \begin{pmatrix} -3 & -2 \\ 3 & 4 \end{pmatrix}$.

a) Find the eigenvalues of A.

$\det(A - \lambda I) = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2)$, so the eigenvalues are 3 and -2.

b) For each eigenvalue, find a corresponding eigenvector.

An eigenvector for $\lambda = 3$ is $(1, -3)$, while one for $\lambda = -2$ is $(2, -1)$.

Problem 5. True of False

a) If A and B are square and invertible, then AB is invertible and $(AB)^{-1} = A^{-1}B^{-1}$. FALSE. $(AB)^{-1} = B^{-1}A^{-1}$

b) If 2 rows of a 3×3 matrix A are the same, then $\det(A) = 0$. TRUE

c) If A is a 4×4 matrix, then $\det(4A) = 4\det(A)$. FALSE. $\det(4A) = 4^4 \det(A)$.

d) If A is invertible, then $\det(A) \neq 0$. TRUE

e) If A is a square stochastic matrix, then $A - I$ is not invertible. TRUE. The steady state solution always satisfies $(A - I)x = 0$.

f) If V is a 5-dimensional vector space, then there is a set of vectors $\{v_1, \ldots, v_6\}$ that spans V. TRUE. Take any basis (5 vectors) and thrown in any extra vector.

g) If V is 5-dimensional, then every collection of 4 vectors in V is linearly independent. FALSE. The set $\{v, 2v, 3v, 4v\}$ isn't linearly independent. There EXIST linearly independent sets of 4 vectors, but not EVERY collection of 4 vectors is linearly independent.

h) If there is a collection $\{v_1, \ldots v_7\}$ that spans a vector space V, then the dimension of V is at least 7. FALSE. The dimension is at MOST 7.

i) If two matrices are row-equivalent, then their null spaces are the same. TRUE. The rows of each matrix are linear combinations of the rows of the other, and hence span the same space.

j) If two matrices are row-equivalent, then their column spaces are the same. FALSE. They have the same dimension, but they are not the same space.