in $\mathbb{R}^n$, $\vec{V} \cdot \vec{W} = \sum_i V_i W_i$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 4 \\ 1 \\ 5 \end{pmatrix} = 1\cdot 3 + 2\cdot 1 + 3\cdot 4 + 4\cdot 1 + 5\cdot 5$$

$$= 46$$

$$|V| = \sqrt{\vec{V}\cdot\vec{V}} = \sqrt{\sum_i V_i^2}$$

Schwartz inequality

For any $\vec{V}, \vec{w}$, $|\vec{V}\cdot\vec{w}| \le |\vec{V}|\,|\vec{w}|$

<u>pf</u> For any t, $|\vec{V} + t\vec{w}|^2 \ge 0$

$$0 \le \min_t \, (\vec{V} + t\vec{w}) \cdot (\vec{V} + t\vec{w})$$

$$= \min_t \left(\vec{V}\cdot\vec{V} + 2t\, \vec{V}\cdot\vec{w} + t^2\, \vec{w}\cdot\vec{w} \right)$$

(Minimum is where $2\vec{V}\cdot\vec{w} + 2t\, \vec{w}\cdot\vec{w} = 0$

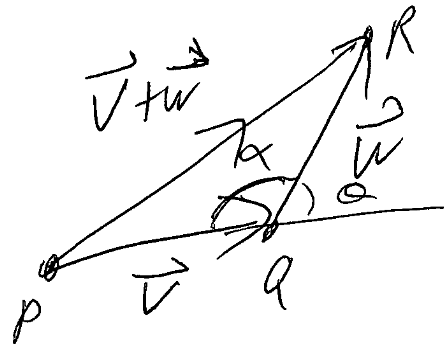
$$t = -\frac{\vec{V}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}$$)

$$0 \le \vec{V}\cdot\vec{V} - 2\frac{(\vec{V}\cdot\vec{w})}{\vec{w}\cdot\vec{w}}(\vec{V}\cdot\vec{w}) + \left(\frac{\vec{V}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)^2 (\vec{w}\cdot\vec{w})$$

$$= \vec{V}\cdot\vec{V} - \frac{(\vec{V}\cdot\vec{w})^2}{(\vec{w}\cdot\vec{w})}$$

Triangle inequality.

$$|\vec{v}+\vec{w}| \le |\vec{v}| + |\vec{w}|$$



$$
\begin{aligned}
\sqrt{|\vec{v}+\vec{w}|^2} &= \sqrt{(\vec{v}+\vec{w})\cdot(\vec{v}+\vec{w})} \\
&= \sqrt{\vec{v}\cdot\vec{v}+\vec{w}\cdot\vec{w}+2\,\vec{v}\cdot\vec{w}} \\
&= \sqrt{|v|^2+|w|^2+2\,\vec{v}\cdot\vec{w}} \\
&\le \sqrt{|v|^2+|w|^2+2|v||w|} \quad \text{(by Schwartz)} \\
&= \sqrt{(|v|+|w|)^2} = |v|+|w|
\end{aligned}
$$

$$\Rightarrow \quad |\vec{v}+\vec{w}| \le |v|+|w|$$

$$
\begin{aligned}
|\vec{v}+\vec{w}|^2 &= |v|^2+|w|^2+2\,\vec{v}\cdot\vec{w} \\
&= |v|^2+|w|^2+2|v||w|\cos\theta \\
&= |v|^2+|w|^2-2|v||w|\cos\alpha
\end{aligned}
$$

$$0 \leq (\vec{V} \cdot \vec{V})(\vec{W} \cdot \vec{W}) - (\vec{V} \cdot \vec{W})^2$$

$$(\vec{V} \cdot \vec{W})^2 \leq (\vec{V} \cdot \vec{V})(\vec{W} \cdot \vec{W})$$

$$|\vec{V} \cdot \vec{W}| \leq |V|\,|W|$$

$\blacksquare$

Def: Angle between $\vec{V}$ and $\vec{W}$ is θ s.t.

$$\cos(\theta) = \frac{\vec{V} \cdot \vec{W}}{|V|\,|W|}$$

$$\theta = \cos^{-1}\left(\frac{\vec{V} \cdot \vec{W}}{|V|\,|W|}\right)$$

$$\vec{V} \cdot \vec{W} = |V|\,|W| \cos(\theta)$$

A unit vector is a vector $\vec{u}$ with $|\vec{u}|=1$ (equivalently, $\vec{u}\cdot\vec{u}=1$)

Given $\vec{v}$, unit vector in $\vec{v}$ direction is

$$\vec{u}=\frac{\vec{v}}{|\vec{v}|}$$

Ex: $\vec{v}=\begin{pmatrix}2\\2\\-1\end{pmatrix}$, $\vec{w}=\begin{pmatrix}2\\-1\\2\end{pmatrix}$

$|\vec{v}|=\sqrt{2^2+2^2+(-1)^2}=\sqrt{9}=3$ $\qquad |\vec{w}|=3$

$$\vec{u}=\frac{\vec{v}}{|\vec{v}|}=\begin{pmatrix}2/3\\2/3\\-1/3\end{pmatrix}$$

$\vec{v}\cdot\vec{w}=|v||w|\cos\theta=2\cdot2+2\cdot(-1)+(-1)\cdot2=0$

$9\cos\theta=0 \qquad \cos\theta=0 \qquad \theta=90^\circ$

---

2 vectors $\vec{v},\vec{w}$ are orthogonal if $\vec{v}\cdot\vec{w}=0$ (angle between them is 90°)

$m \times n$

Matrix $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{pmatrix}$

vector $x \in \mathbb{R}^n$. $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$Ax = x_1 \vec{a}_1 + \cdots + x_n \vec{a}_n$$

$$(Ax)_i = (x_1 \vec{a}_1 + \cdots x_n \vec{a}_n)_i$$

$$= x_1 a_{i1} + \cdots x_n a_{in} = \sum_{j=1}^{n} a_{ij} x_j$$

$$= (i\text{-th row of } A) \begin{pmatrix} x \end{pmatrix}$$

Ex: $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 7\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$$= \begin{pmatrix} 1\cdot 3 + 1\cdot 1 + 1\cdot 7 \\ 1\cdot 3 + 2\cdot 1 + 3\cdot 7 \end{pmatrix} = \begin{pmatrix} 11 \\ 26 \end{pmatrix}$$

If A is $m \times n$ matrix

and B is $n \times p$ matrix $= (\vec{b}_1 \cdots \vec{b}_p)$

$$AB = (A\vec{b}_1, \cdots\cdots A\vec{b}_p)$$

formula for matrix mult.

$$(AB)\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix} = \text{First column of } AB$$

$$= A\vec{b}_1 = A\left(B\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}\right)$$

$$\begin{tikzcd}
\mathbb{R}^m & & \mathbb{R}^p \arrow[ll, "AB"'] \arrow[dl, "B"] \\
& \mathbb{R}^n \arrow[ul, "A"] &
\end{tikzcd}$$

Concept of matrix mult.

$$(AB)_{ik} = \sum_{j=1}^{n} A_{ij} \underbrace{B_{jk}}_{(\vec{b}_k)_j}$$

More useful formula

$$(AB)C = A(BC) \qquad (ABC)_{il} = \sum_{j,k} A_{ij} B_{jk} C_{kl}$$

1) Write $\begin{pmatrix} 7 \\ 15 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

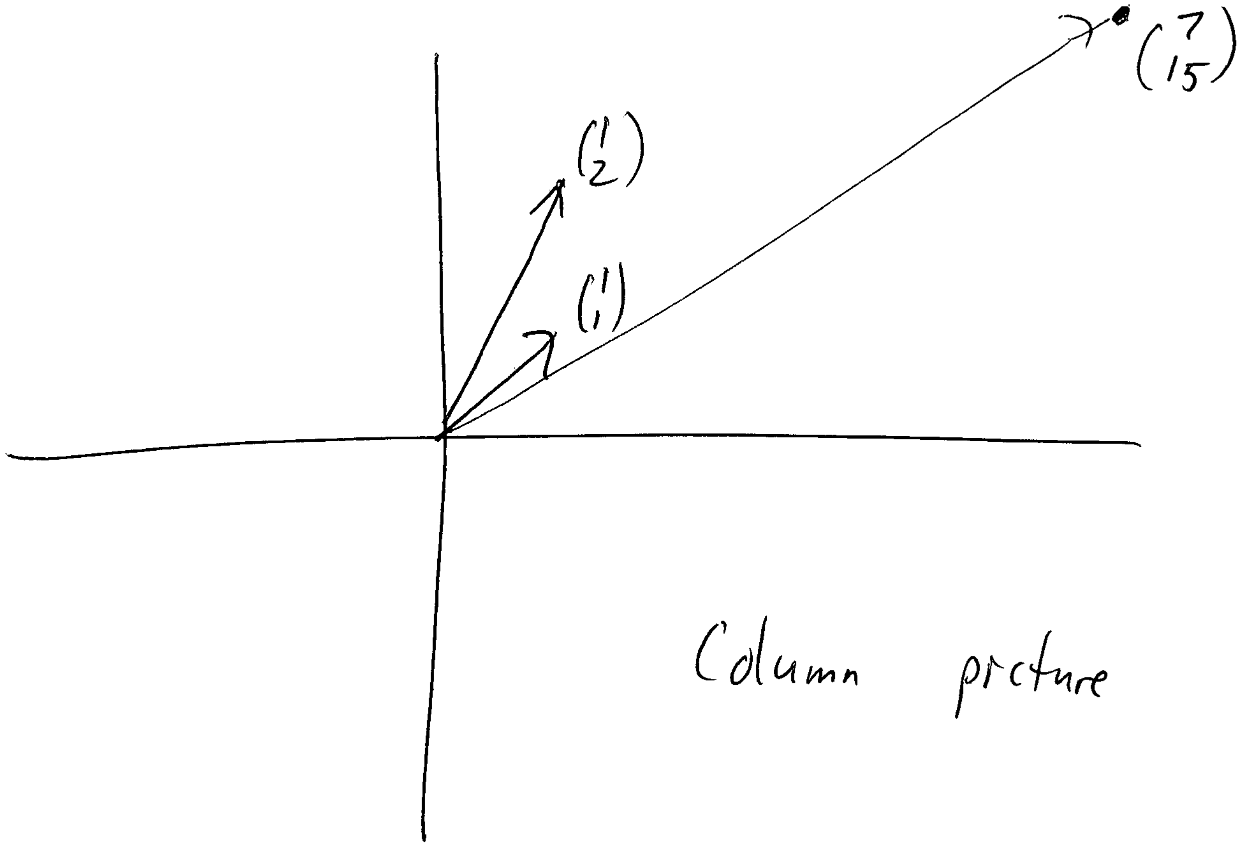
2) Solve $x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 15 \end{pmatrix}$

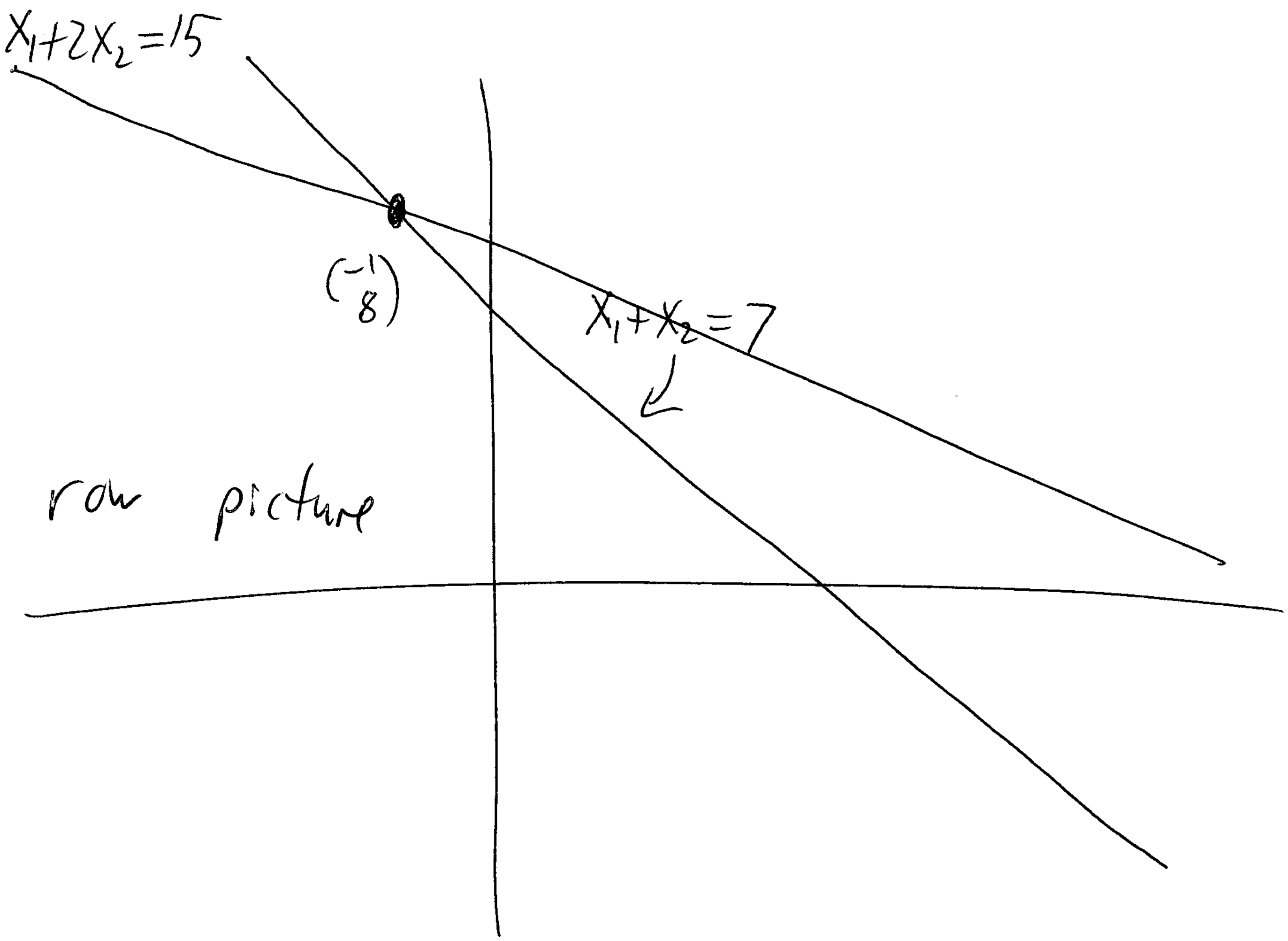
3) Solve $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 15 \end{pmatrix}$

4) Solve
$$x_1 + x_2 = 7$$
$$x_1 + 2x_2 = 15$$

$\left(\vec{x} = \begin{pmatrix} -1 \\ 8 \end{pmatrix} \right)$

$\boxed{\text{ALL THE SAME}}$


$\binom{7}{15}$
$\binom{1}{2}$
$\binom{1}{1}$
Column picture


$x_1+2x_2=15$
$\binom{-1}{8}$
$x_1+x_2=7$
row picture

$$x_1 + x_2 = 7$$
$$x_1 + 2x_2 = 15$$

$$\left(\begin{array}{cc|c} 1 & 1 & 7 \\ 1 & 2 & 15 \end{array}\right)$$ Augmented matrix

---

$$x_1 + x_2 = 7$$
$$x_2 = 8$$

$$\left(\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 1 & 8 \end{array}\right)$$

$$x_1 = 7 - x_2 = -1$$

$$x_2 = 8$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 1 & 2 & 3 & 8 \\ 1 & 4 & 8 & 9 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 7 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\left.\begin{aligned} x_1 &= 7 - x_2 - x_3 = 5 \\ x_2 &= 1 - 2x_3 = 3 \\ x_3 &= -1 \end{aligned}\right\} \text{Back Substitution}$$

What about $\begin{pmatrix}1&1\\1&2\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}b_1\\b_2\end{pmatrix}$?

$$x_1 + x_2 = b_1$$
$$x_1 + 2x_2 = b_2$$
$$\Downarrow$$
$$x_2 = b_2 - b_1$$
$$\Downarrow$$
$$x_1 + (b_2 - b_1) = b_1 \implies x_1 = 2b_1 - b_2$$

$$\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}2&-1\\-1&1\end{pmatrix}\begin{pmatrix}b_1\\b_2\end{pmatrix}$$

Note: $\begin{pmatrix}2&-1\\-1&1\end{pmatrix}\begin{pmatrix}1&1\\1&2\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$

$\begin{pmatrix}1&1\\1&2\end{pmatrix}\begin{pmatrix}2&-1\\-1&1\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$.

<u>Thm</u>: If A has an inverse, then $Ax=b$ has a <u>unique</u> solution, namely ~~$b=A$~~ x. $x = A^{-1}b$.

$$\left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 1 & 2 & b_2 \end{array}\right)$$

$$x_1 + x_2 = b_1$$
$$x_1 + 2x_2 = b_2$$

; $x_1\begin{pmatrix}1\\1\end{pmatrix} + x_2\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}b_1\\b_2\end{pmatrix}$

$$\left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & b_2-b_1 \end{array}\right)$$

$$x_1 = b_1 - x_2$$
$$x_2 = b_2 - b_1$$
$$x_1 = b_1 - (b_2 - b_1) = 2b_1 - b_2$$

$$\left.\begin{array}{l} x_1 = 2b_1 - b_2 \\ x_2 = b_2 - b_1 \end{array}\right\} \quad \begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}2 & -1\\-1 & 1\end{pmatrix}\begin{pmatrix}b_1\\b_2\end{pmatrix}$$

Def: The inverse of a matrix A is a matrix A^{-1} s.t. $AA^{-1} = A^{-1}A = I$

$$= \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & \cdots & \\ 0 & & \ddots & 0 \\ & & & 1 \end{pmatrix}$$

$$\begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}\begin{pmatrix}2 & -1\\-1 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \checkmark$$

$$\begin{pmatrix}2 & -1\\-1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 2\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \checkmark$$

Thm

If A^{-1} exists then there is exactly one solution to

$$Ax = b, \text{ namely } x = A^{-1}b.$$

Note: Only some square matrices have inverses (and no rectangular).

$$Ax = A(A^{-1}b) = (AA^{-1})b = Ib = b \checkmark$$

solution works

If $Ax = b$, then

$$A^{-1}Ax = A^{-1}b$$

$$Ix = A^{-1}b$$

$$x = A^{-1}b \checkmark$$

solution is unique.