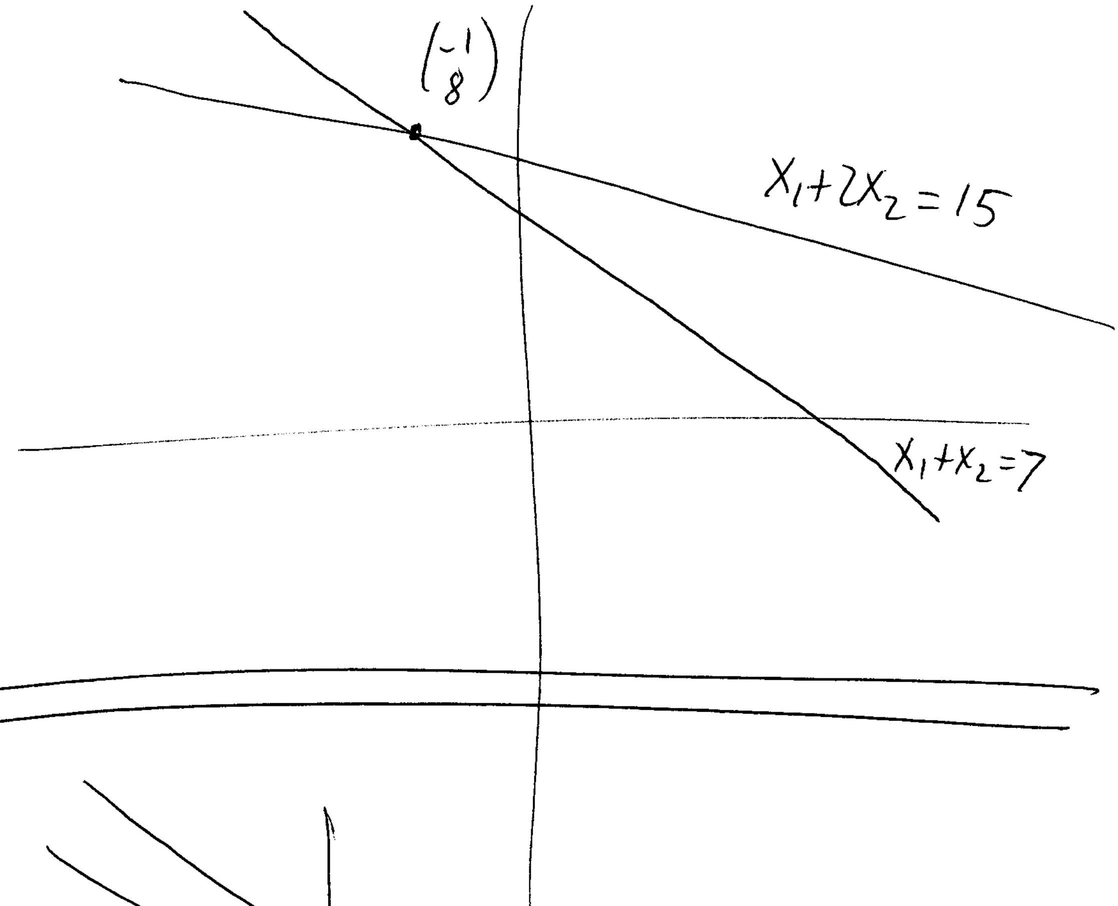
Solving n equ's in n unknowns

Start with 2x2.

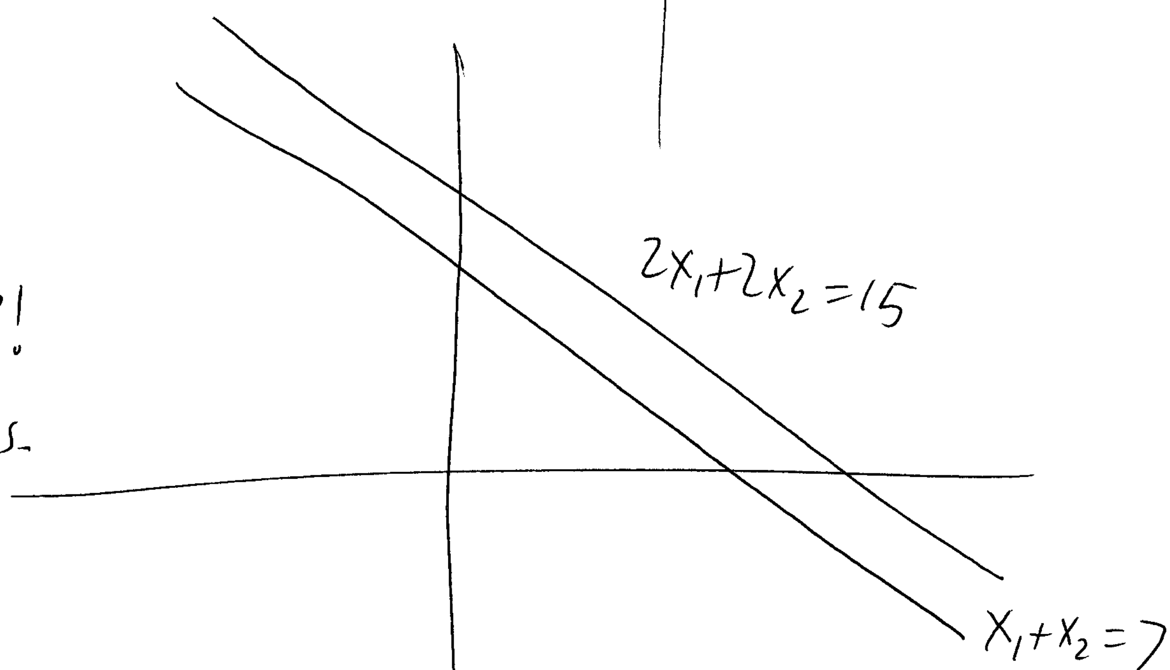
$$\left(\begin{array}{cc|c} 1 & 1 & 7 \\ 1 & 2 & 15 \end{array}\right)$$

$x_1 + x_2 = 7$
$x_1 + 2x_2 = 15$



$x_1 + x_2 = 7$
$2x_1 + 2x_2 = 15$
$\Downarrow$
$0 = 1$?!

No solutions.

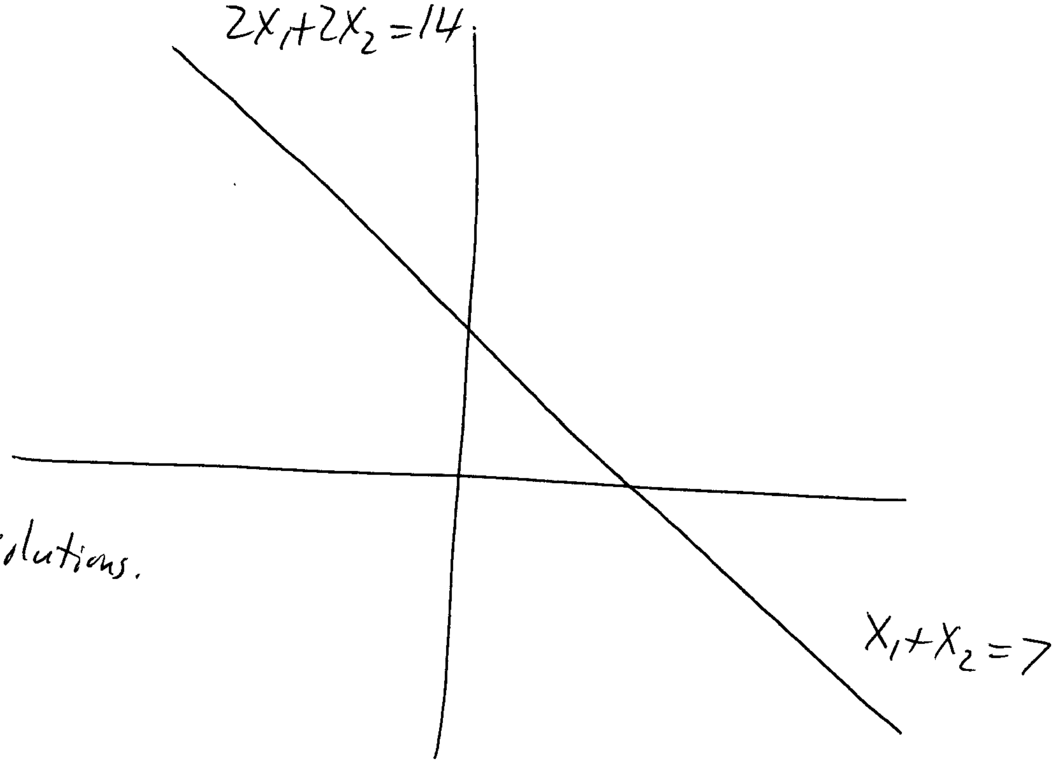


$$X_1 + X_2 = 7$$
$$2X_1 + 2X_2 = 14$$

$$\Downarrow$$

$$0 = 0$$

infinitely many solutions.



In general, the # of solutions to $Ax=b$ is $0, 1,$ or ∞.

To solve n equ's in n unknowns.

1) Write system $Ax=b$ as augmented matrix

$[A \mid b]$

2) Apply row operations to simplify this.

3) Read off solutions from simplified matrix.

Row operations:

1) Add a multiple of one row to another.
2) Swap two rows.
3) Rescale a row by a <u>nonzero</u> amount.

What are the desired forms.

1) Row-echelon form (REF)

2) Reduced row-echelon form (RREF)

A matrix is in REF if

1) Any row consisting of zeroes is at the bottom.

2) For any other row, the first non-zero entry (pivot) is to the right of the previous pivot.

E.g. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 3 & 5 & 9 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 15 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ is in REF

$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 3 & 5 \end{pmatrix}$ Not in REF

How to get REF (typical case)

1) Subtract a multiple of 1st row from others to clear out the 1st column.
2) Then subtract multiples of 2nd row from 3rd, 4th ... rows to clear out 2nd col.
3) Repeat as many times as needed.

Ex: $\left(\begin{array}{cc|c} 2 & 3 & 5 \\ 4 & 9 & 13 \end{array}\right) \xrightarrow[r_2 \to r_2 - 2r_1]{} \left(\begin{array}{cc|c} 2 & 3 & 5 \\ 0 & 3 & 3 \end{array}\right)$

REF

Ex: $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 6 \\ 1 & 4 & 8 & 13 \end{array}\right) \xrightarrow[\substack{r_2 \to r_2 - r_1 \\ r_3 \to r_3 - r_1}]{} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 3 & 7 & 10 \end{array}\right)$

$\xrightarrow[r_3 \to r_3 - 3r_2]{} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right)$ REF.

$$\left(\begin{array}{ccc|c} 1&1&1&3\\ 1&1&2&4\\ 1&2&2&5 \end{array}\right) \to \left(\begin{array}{ccc|c} 1&1&1&3\\ 0&0&1&1\\ 0&1&1&2 \end{array}\right) \xrightarrow[r_3 \leftrightarrow r_2]{} \left(\begin{array}{ccc|c} 1&1&1&3\\ 0&1&1&2\\ 0&0&1&1 \end{array}\right)$$

1 soln.

4) If leading entry of a row is 0, swap rows!

What if whole colum is 0?

5) Go to next column.

$$\left(\begin{array}{ccc|c} 1&1&1&3\\ 1&1&5&7\\ 1&1&7&9 \end{array}\right) \to \left(\begin{array}{ccc|c} 1&1&1&3\\ 0&0&4&4\\ 0&0&6&6 \end{array}\right)$$

$r_3 \to r_3 - \frac{3r_2}{2}$

$$\left(\begin{array}{ccc|c} 1&1&1&3\\ 0&0&4&4\\ 0&0&0&0 \end{array}\right)$$

∞ many soln.

$$\left(\begin{array}{ccc|c} 1&1&1&3\\ 1&1&5&7\\ 1&1&7&10 \end{array}\right) \to \left(\begin{array}{ccc|c} 1&1&1&3\\ 0&0&4&4\\ 0&0&6&7 \end{array}\right) \to \left(\begin{array}{ccc|c} 1&1&1&3\\ 0&0&4&4\\ 0&0&0&1 \end{array}\right)$$

No soln.

What to do with REF

Typical: REF is triangular.

Ex:
$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \end{array}\right)$$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 5x_2 + 6x_3 &= 7 \\ 8x_3 &= 9 \end{aligned} \qquad \begin{aligned} x_1 &= 4 - 2x_2 - 3x_3 \\ x_2 &= (7 - 6x_3)/5 \\ x_3 &= 9/8 \end{aligned}$$

Each row describes one variable in terms of subsequent variables.

Start at bottom, substitute in previous equations to get all varrabbs

"Back Substitution" works bottom → top
right → left

What if there's a row of 0's in coefficients?

If a row is $0=0$, ignore it. (no info)

If it's $0 = $ (something other than 0), contradiction! (no soln)

Ex: $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & 4 & 4 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right)$

$X_1 = 3 - X_2 - X_3$

$X_2 =$ arbitrary.

$X_3 = 1$

$0 = 0$ (ignore)

X_2 = free variable.

All variables are either pivot variables or free variables.

Free variables are <u>arbitrary</u>.

Pivot variables are then determined.

Summary of $n \times n$ case.

1) If REF is triangular (typical), $\exists !$ solution

$\exists$: exists

$!$: unique

2) If REF has row w/ contradictions, no soln.

3) If REF has row(s) of $0=0$ & no contradictions, ∞ many solutions. Free variables are free, and they determine pivot variables

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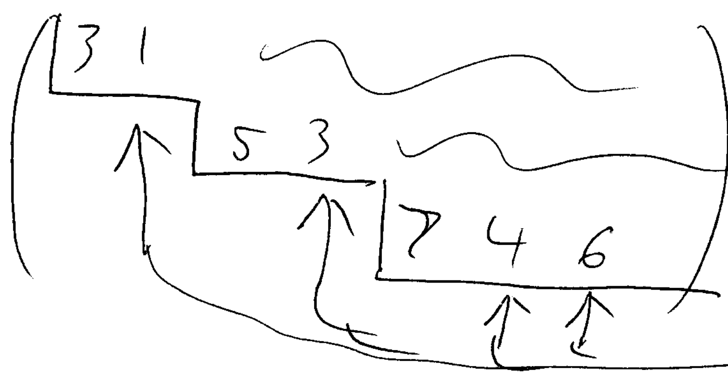
~~Fam~~ In ~~m~~ $m \times n$ case w/ $m < n$ (more variables than equations), always have free variables. At most m pivots, At least $n-m$ free variables.

Either no soln or infinitely many.



free variables

What ~~is~~ ~~0~~. if $m>n$?

(more equ. that variables)

Always have row of 0's on LHS.

# of pivots at most n.

At least $m-n$ rows of 0's.