Vector Spaces

<u>Def</u> A <u>vector space</u> is a set where
1) You can add elements.
2) You can multiply elements by scalars.
3) The usual rules of arithmetic apply

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8 "usual rules"

1) $\vec{x}+\vec{y} = \vec{y}+\vec{x}$ Addition is commutative
2) $\vec{0}+\vec{x} = \vec{x}$ Additive identity
3) $(-\vec{x})+\vec{x} = \vec{0}$ Additive inverse
4) $1\vec{x} = \vec{x}$ multiplicative identity
5) $a(b\vec{x}) = (ab)\vec{x}$ Associative multiplication
6) $(\vec{x}+\vec{y})+\vec{z} = \vec{x}+(\vec{y}+\vec{z})$ Associative addition
7) $a(\vec{x}+\vec{y}) = a\vec{x}+a\vec{y}$
8) $(a+b)\vec{x} = a\vec{x}+b\vec{x}$

$\Big\}$ Distributive laws.

Examples of vector spaces.

1) $\mathbb{R}^n$. - lists of #s.

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{pmatrix}$$

$$c \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} cx_1 \\ \vdots \\ cx_n \end{pmatrix}$$

2) All functions of a variable t.

3) All equations.

4) ~~Polyn~~ Cubic polynomials $a_0 + a_1 t + a_2 t^2 + a_3 t^3$

5) All 3x5 matrices

6) And many more.

7) Certain subsets of existing vector spaces (especially $\mathbb{R}^n$)

<u>Def</u> Let V be a vector space.
A subset $S \subset V$ is a <u>subspace</u> if it is a vector space in its own right. (using $+, \times$ from V)

<u>Thm</u> S is a subspace if it meets 3 conditions

a) Closed under addition. If $\vec{x}, \vec{y} \in S$, so is $\vec{x}+\vec{y}$.

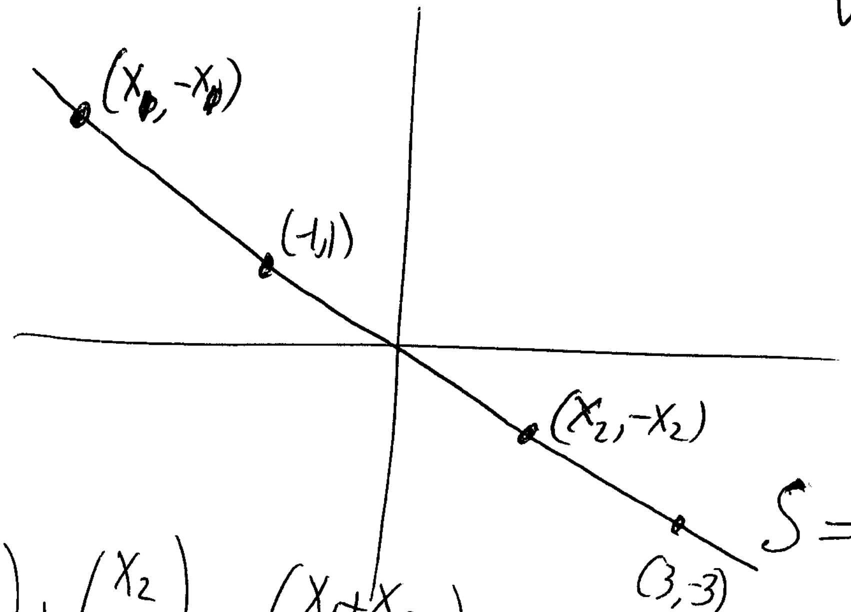
b) Closed under scalar mult. If $\vec{x} \in S$, so is $c\vec{x}$.

c) S is non-empty

(can also replace (c) with $\vec{0} \in S$)

Ex 1

$V = \mathbb{R}^2$



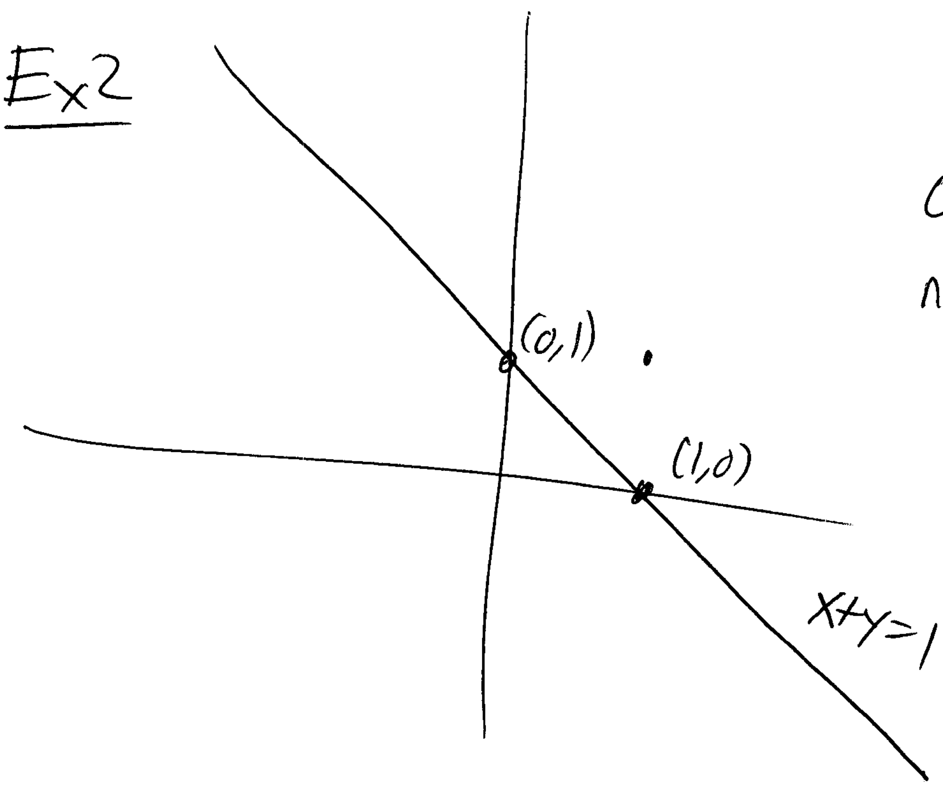
$S = \{x+y=0\}$

$$\begin{pmatrix} x_1 \\ -x_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ -x_2 \end{pmatrix} = \begin{pmatrix} x_1+x_2 \\ -x_1-x_2 \end{pmatrix} \in S \checkmark$$

$$c\begin{pmatrix} x_1 \\ -x_1 \end{pmatrix} = \begin{pmatrix} cx_1 \\ -cx_1 \end{pmatrix} \in S \checkmark$$

S is subspace

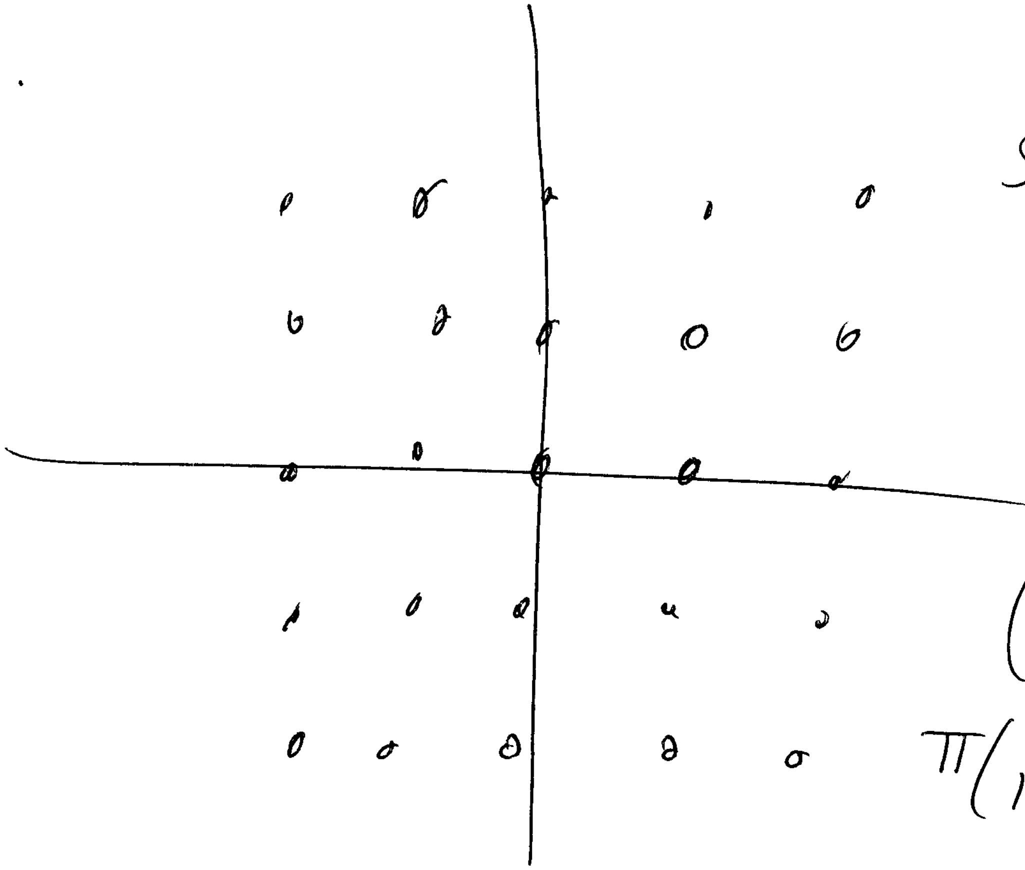
Ex 2



0 for 3

not a subspace.

Ex3.



$S = \mathbb{Z}^2$

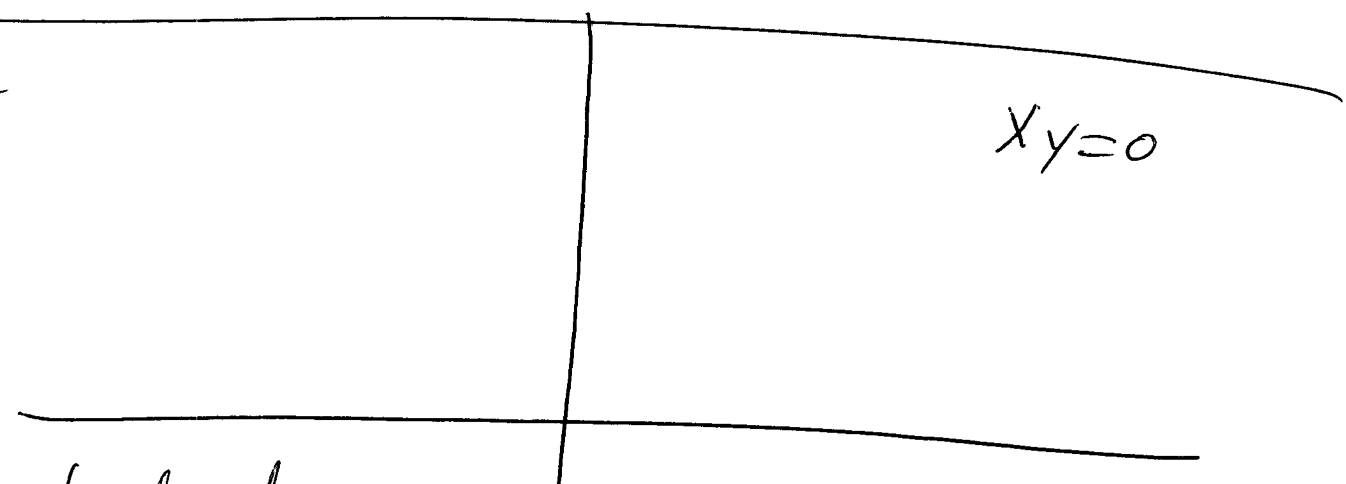
$\begin{pmatrix}1\\1\end{pmatrix} \in \mathbb{Z}^2$, but

$\pi\begin{pmatrix}1\\1\end{pmatrix} \notin \mathbb{Z}^2$.

Not closed under scalar mult.
Not a subspace.

Ex4

$xy = 0$

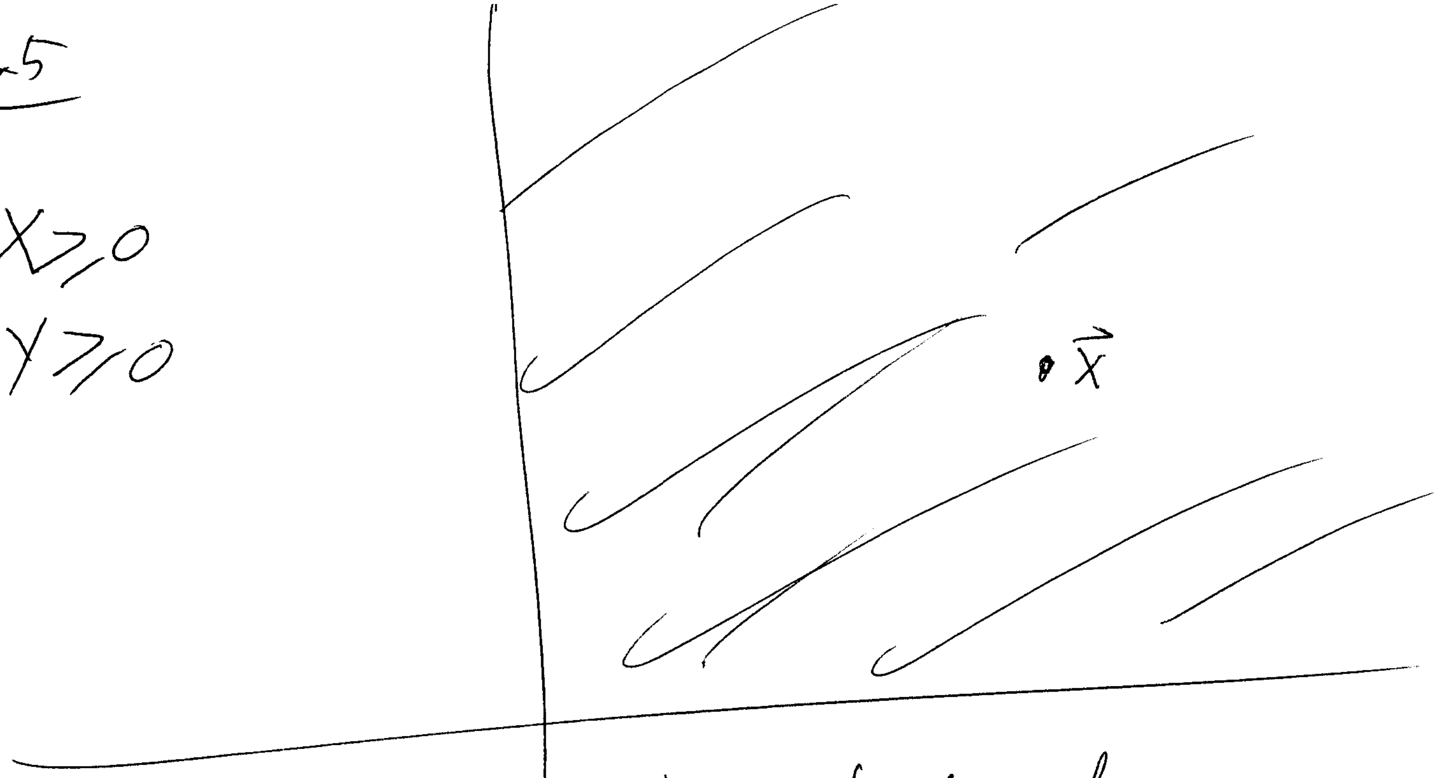


Not closed under addition.
$\begin{pmatrix}3\\0\end{pmatrix} + \begin{pmatrix}0\\5\end{pmatrix} \notin$ set.

Ex 5

$x \geq 0$

$y \geq 0$



$\vec{x} \in S$

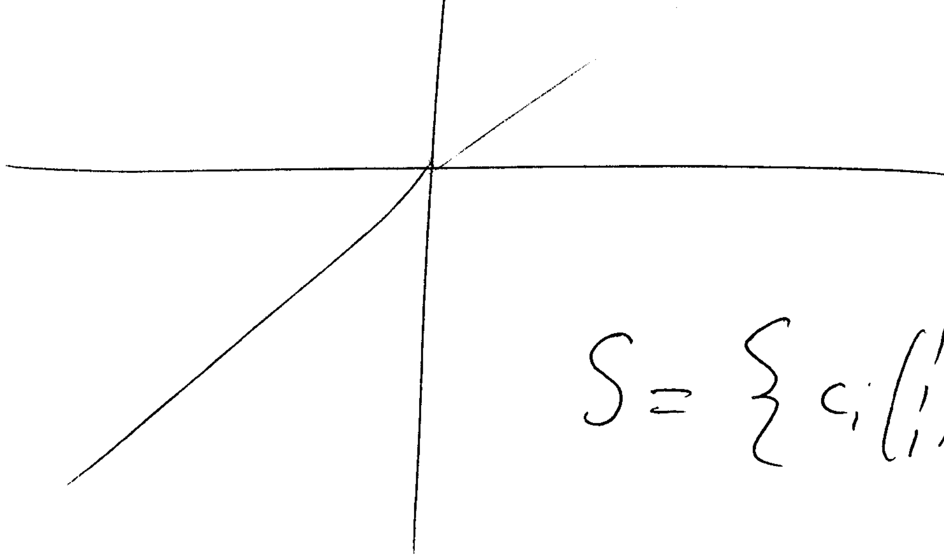
$-3\vec{x} \notin S$

Not closed under mult. by negative scalars.

Not a subspace.

Ex 6 In $\mathbb{R}^3$,

$S =$ all linear comb of $\begin{pmatrix}1\\1\\1\end{pmatrix}$ and $\begin{pmatrix}2\\4\\3\end{pmatrix}$



$$S = \left\{ c_1 \begin{pmatrix}1\\1\\1\end{pmatrix} + c_2 \begin{pmatrix}2\\4\\3\end{pmatrix} \right\}.$$

$$\vec{x} = c_1\begin{pmatrix}1\\1\\1\end{pmatrix} + c_2\begin{pmatrix}2\\4\\3\end{pmatrix}$$

$$\vec{y} = c_3\begin{pmatrix}1\\1\\1\end{pmatrix} + c_4\begin{pmatrix}2\\4\\3\end{pmatrix}$$

$$\vec{x} + \vec{y} = (c_1+c_3)\begin{pmatrix}1\\1\\1\end{pmatrix} + (c_2+c_4)\begin{pmatrix}2\\4\\3\end{pmatrix}$$

$$\in S$$

$$a\vec{x} = ac_1\begin{pmatrix}1\\1\\1\end{pmatrix} + ac_2\begin{pmatrix}2\\4\\3\end{pmatrix} \in S$$

Let $\{\vec{v}_1, \dots \vec{v}_k\}$ be vectors in V

Span $\{\vec{v}_1, \dots \vec{v}_k\} := \{$all linear combinations of $\vec{v}_1, \dots \vec{v}_k\}$.

Claim: Span $\{\vec{v}_1, \dots \vec{v}_k\}$ is a subspace, no matter what $\vec{v}_1, \dots \vec{v}_k$ are.

$$\vec{x} = a_1\vec{v}_1 + \cdots + a_k\vec{v}_k$$

$$\vec{y} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$$

$$\vec{x}+\vec{y} = (a_1+c_1)\vec{v}_1 + \cdots + (a_k+c_k)\vec{v}_k \in \text{Span}\{\vec{v}_1, \ldots \vec{v}_k\}$$

$$b\vec{x} = ba_1\vec{v}_1 + \cdots + ba_k\vec{v}_k \in \text{Span}\{\vec{v}_1, \ldots \vec{v}_k\}$$

<u>Def</u>: If A is an $m \times n$ matrix,

$\text{Col}(A) = \text{Span}(\text{columns of } A)$

$= \text{subspace of } \mathbb{R}^m$.

In $\mathbb{R}^3$, line parallel to $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$$

In $\mathbb{R}^3$, Span $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$ is plane $x+y-2z=0$

$$\text{Col}\begin{pmatrix} 1 & 2 \\ 1 & 4 \\ 1 & 3 \end{pmatrix} = \text{Nul}\begin{pmatrix} 1 & 1 & -2 \end{pmatrix}$$

$$\mathrm{Nul}(A) = \{\vec{x} \mid A\vec{x} = 0\}$$

$$= \text{Subspace of } \mathbb{R}^n$$

$A\vec{x} = 0$

$A\vec{y} = 0$

$A(\vec{x}+\vec{y}) = \vec{0}+\vec{0} = \vec{0}$

$A(c\vec{x}) = c\,A\vec{x} = c\,\vec{0} = \vec{0}.$

$A\vec{0} = \vec{0}.$