A vector space is a set where

1) You can take linear combinations
2) Arithmetic works.

Many examples, like $\mathbb{R}^n$, spaces of functions, spaces of matrices, spaces of equations, etc.

If V is a vector space, and $W \subset V$ is a subspace, then

1) W is closed under addition.
2) W is closed under scalar multiplication
3) $\vec{0} \in W$

<u>Thm</u> If $W \subset V$ satisfies 1-3, then it is a subspace.

How to make a subspace (of $\mathbb{R}^n$)

1) Build up. Start with $\vec{0}$, and add vectors. $\beta = \{\vec{v}_1, \dots \vec{v}_k\}$ $\quad \vec{v}_i \in V$.

$$\text{Span}(\beta) = \{\text{all linear combinations of } \vec{v}_1, \dots \vec{v}_k\}.$$

$$\text{Span}(\emptyset) = \{\vec{0}\}.$$

What does span (bunch of vectors in $\mathbb{R}^3$) look like?

$$\text{Either: } \begin{cases} \vec{0} \\ \text{line} \\ \text{plane} \\ \text{all of } \mathbb{R}^3 \end{cases}$$

$$A = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_k \end{pmatrix}$$

$\mathrm{Col}(A) = \mathrm{Span}(\text{columns of } A)$

$= \text{All liner combinations of columns}$

$= \{A\vec{x}\}$

$= \{\text{all } \vec{b} \text{ such that } A\vec{x} = \vec{b} \text{ has a solution}\}$

Cut down:

Start with all of V and apply linear conditions.

Ex: $V = \mathbb{R}^3$, $\quad W = \begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 + 2x_2 + 3x_3 = 0 \\ 3x_1 + 5x_2 + 7x_3 = 0 \\ 7x_1 + 11x_2 + 15x_3 = 0 \end{cases}$

All solutions to $Ax = 0$, where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 5 & 7 \\ 7 & 11 & 15 \end{pmatrix}$$

For any matrix A, $\mathrm{Nul}(A) = \{\vec{x} \mid A\vec{x} = 0\}$

$= $ all solutions to $A\vec{x} = 0$

$= $ all ways to write 0 as a linear comb of columns of A.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$$

$$\mathrm{Col}(A) = \mathrm{Span}\left\{ \overset{\vec{u}}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}, \overset{\vec{v}}{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}} \right\} = \text{plane in } \mathbb{R}^3$$

$$= \{ c_1 \vec{u} + c_2 \vec{v} \}$$

$\mathrm{Nul}(A) =$ all solutions to

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_1 + 2x_2 &= 0 \\ x_1 + 3x_2 &= 0 \end{aligned}$$

$=$ subspace of $\mathbb{R}^2$.

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ 0 &= 0 \end{aligned}$$

$$\mathrm{Nul}(A) = \{ \vec{0} \in \mathbb{R}^2 \}$$

If A is $m \times n$, $\mathrm{Col}(A) \subset \mathbb{R}^m$

$\mathrm{Nul}(A) \subset \mathbb{R}^n$

$B = \begin{pmatrix} 1 & -2 & 1 \end{pmatrix}$

$Col(B) =$ all of $\mathbb{R}^1$

$Nul(B) = \{x_1 - 2x_2 + x_3 = 0\} =$ plane in $\mathbb{R}^3$

$= Col(A)$

$A^T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$ $\qquad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$

$Nul(A^T)$

$x_1 = x_3$

$x_2 = -2x_3$

$x_3 = x_3$

$\vec{x} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$Nul(A^T) =$ line in $\mathbb{R}^3$ $=$ all multiples of $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$= Col(B^T)$

Four fundamental subspaces associated to an $m \times n$ matrix A.

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1) $\mathrm{Col}(A) \subset \mathbb{R}^m$ Column space

2) $\mathrm{Nul}(A) \subset \mathbb{R}^n$ Null space or Kernel

3) $\mathrm{Col}(A^T) \subset \mathbb{R}^n$ Row space.

4) $\mathrm{Nul}(A^T) \subset \mathbb{R}^m$ Cokernel.

How to solve $Ax = 0$.

1) Put A in REF.

2) Put A in RREF

3) Identify free & pivot variables.

4) Find special solutions

    Set one free variable $= 1$

        other " " $= 0$

    Solve for pivot variables.

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \text{ REF}$$

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix} \text{ RREF}$$

$$\left.\begin{aligned} X_1 \quad\quad - X_3 - 2X_4 &= 0 \\ X_2 + 2X_3 + 3X_4 &= 0 \end{aligned}\right\} \Longrightarrow \begin{aligned} X_1 &= X_3 + 2X_4 \\ X_2 &= -2X_3 \cancel{+} -3X_4 \\ X_3 &= X_3 \\ X_4 &= \quad\quad X_4 \end{aligned}$$

$$\text{Special soln: } \quad \overset{\vec{u}}{\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}}, \; \overset{\vec{v}}{\begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}}$$

$$\vec{X} = X_3 \vec{u} + X_4 \vec{V}.$$

$$\text{Null}(A) = \text{Span}\{\vec{u}, \vec{v}\}.$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 5 & 7 \\ 7 & 11 & 15 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 4 & 8 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$X_1 = X_3$$
$$X_2 = -2X_3$$
$$X_3 = X_3$$

$$\vec{X} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \vec{X_3}$$

$\text{Rank}(A) = \#$ of pivots.

If A is $m \times n$ matrix with rank K.

K pivot variables

$n-K$ free variables

$n-K$ special solutions to $Ax=0$.