V is a vector space

$\mathcal{B} = \{\vec{b}_1, \dots \vec{b}_k\}$

<u>Def</u> $\mathcal{B}$ is linearly independent if the only way to get $0 = a_1\vec{b}_1 + \dots + a_k\vec{b}_k$ is with $a_1 = a_2 = \dots a_k = 0$.

If $V = \mathbb{R}^n$, look at $B = (\vec{b}_1 \cdots \vec{b}_k)$

$\mathcal{B}$ is linearly independent $\Longleftrightarrow$ rref(B) has a pivot in each column.

~~Def: $\mathcal{B}$ spans V if {all Span~~

Def: $\text{Span}(\mathcal{B}) = \{$all linear combinations of $\vec{b}_1 \dots \vec{b}_k\}$

Def: $\mathcal{B}$ spans V if $\text{Span}(\mathcal{B}) = V$

If $V = \mathbb{R}^n$

$\mathcal{B}$ spans $\mathbb{R}^n \iff$ every vector in $\mathbb{R}^n$ is a lin comb of $\vec{b}_1, \dots \vec{b}_k$

$\iff$ For every $\vec{c} \in \mathbb{R}^n$, you can solve $\mathcal{B}\vec{x} = \vec{c}$

$\iff$ rref($\mathcal{B}$) has a pivot in each row.

In $\mathbb{R}^3$, $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 10 \end{pmatrix} \right\} = \mathcal{B}$

$$\mathcal{B} = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 5 & 7 \\ 3 & 6 & 7 & 10 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Not lin ind (since $2 < 4$)

Doesn't span (since $2 < 3$)

<u>Def</u> A <u>basis</u> for V is a set $\mathcal{B}$ that is lin ind and spans.

<u>Thm</u> Every basis $\mathcal{B}$ for $\mathbb{R}^n$ consists of exactly n vectors.

<u>Pf</u>: If $>n$, linearly dependent.
If $<n$ doesn't span.

<u>Thm</u> Any collection of n vectors in $\mathbb{R}^n$ is either
1) A basis, or $\Leftarrow$ n pivots
2) Neither lin ind nor spans. $\Leftarrow$ $<n$ pivots.

Thm Let V be a vector space with a basis $\mathcal{B} = \{\vec{b}_1, \dots \vec{b}_n\}$. Let $\mathcal{C} = \{\vec{c}_1, \dots \vec{c}_m\}$ be another basis. Then $m = n$.

Pf ~~Since B spa~~ Suppose $m > n$.

Since $\mathcal{B}$ spans,

$$\cancel{\vec{b}_1 = a_{11}\vec{c}_1 + \dots + a_{1m}\vec{c}_m}$$

$$\vec{c}_1 = a_{11}\vec{b}_1 + \dots + a_{1n}\vec{b}_n$$

$$\vec{c}_2 = a_{21}\vec{b}_1 + \dots + a_{2n}\vec{b}_n$$

$$\vdots$$

$$\vec{c}_m = a_{m1}\vec{b}_1 + \dots + a_{mn}\vec{b}_n$$

Row-reduce to get

$$\left(\begin{matrix}\text{some}\\ \text{comb of } \vec{c}\text{'s}\end{matrix}\right) = 0\vec{b}_1 + \dots + 0\vec{b}_n = \vec{0}$$

So $\mathcal{C}$ isn't linearly independent!

Def The <u>dimension</u> of V =

$\#$ of vectors in a basis for V.

Ex 1: $V = \mathbb{R}^n$ $\quad \mathcal{B} = \{\vec{e}_1, \vec{e}_2, \ldots \vec{e}_n\}$, where

$$\vec{e}_k = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow k\text{th slot.}$$

$\vec{e}_k$ = k-th column of I.

$\dim \mathbb{R}^n = n$.

Ex 2 : $V = \mathbb{P}_2[t] = \{$quadratic polynomials in $t\}$.

$\mathcal{B} = \{1, t, t^2\}$.

$\dim V = 3$

Ex 3: $V = M_{22} = \{2\times 2$ matrices$\}$.

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

$\dim V = 4$.

$V =$ Solutions to $\frac{d^2f}{dx^2} - f = 0$

$\mathcal{B} = \{e^x, e^{-x}\}$ $\qquad$ $\dim V = 2.$

A is an $m \times n$ matrix of rank r.

m rows
n columns

rref(A) has r pivots.

$V_1 = Col(A) = Span(\text{Columns}) = \{Ax\}.$

Free columns are lin comb of pivot cols, so
$Col(A) = Span(\text{pivot columns})$
pivot columns are lin ind.
Basis $= \{\text{pivot columns of } A\}.$
dimension $Col(A) = r$

$V_2 = Nul(A) =$ Solutions to $\{Ax=0\}$.

Basis $= \{$ special solutions $\}$

dim $=$ # special $=$ # free variables

$= n-r.$

$Nul(A)$ is $n-r$ dimensional subspace of $\mathbb{R}^n$

($Col(A)$ is r-diml subspace of $\mathbb{R}^m$)

$V_3 = Row(A) = Col(A^T)$

$Row(A) = Row(rref(A))$

Basis $= \{(\text{non-zero rows of } rref(A))^T\}$

dim $Row(A) =$ # pivots $= r$.

$Rank(A) = Rank(A^T)$

$V_4 = Nul(A^T)$ $\qquad$ $\dim(V_4) = m - rank(A^T)$

$= m-r$

Basis $=$ a little more complicated.

$=$ special soln to $A^Tx=0$.

Col space is r-diml subspace of $\mathbb{R}^m$

Basis = pivot cols.

Row space is r-diml subspace of $\mathbb{R}^n$

Basis = (nonzero) rows of rref

orthogonal

Nul space is n-r diml subspace of $\mathbb{R}^n$

Basis = special soln to $Ax=0$.

$Nul(A^T)$ is m-r diml subspace of $\mathbb{R}^m$

Basis = special soln to $A^Tx=0$.

Orthogonal.

$$A = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 5 & 7 \\ 3 & 6 & 7 & 10 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \text{rref}(A)$$

$$\text{Basis for } Col(A) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix} \right\} = \text{pivot cols of } A$$

$$r = 2$$

Nul A:

$$\begin{aligned} x_1 &= -2x_2 - x_4 \\ x_2 &= x_2 \\ x_3 &= -x_4 \\ x_4 &= x_4 \end{aligned}$$

$$\text{Basis for } Nul(A) = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

$$\cancel{A=}\ n - r = 4 - 2 = 2$$

$$\text{Basis for } Row(A) = Row(\text{rref}(A)) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$r = 2$$

orthogonal!