$A = LU$ <u>usually</u> works, but not always.

$$\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} b & c \\ 0 & d \end{pmatrix}$$

$$= \begin{pmatrix} b & \sim \\ ab & \sim \end{pmatrix}$$

Can't have $b = 0$, $ab = 1$.

Sometimes need to swap rows.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} - & r_1 & - \\ - & r_2 & - \end{pmatrix} = \begin{pmatrix} - & r_2 & - \\ - & r_1 & - \end{pmatrix}$$

A <u>permutation matrix</u> changes the order of rows.

= Identity matrix w/ rows scrambled.

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} - r_1 - \\ - r_2 - \\ - r_3 - \end{pmatrix} = \begin{pmatrix} - r_2 - \\ - r_3 - \\ - r_1 - \end{pmatrix}$$

A permutation matrix is mostly zeroes, but with one 1 in each column
one 1 in each row

If P_1, P_2 are permutation matrices, so are $P_1 P_2$, $P_2 P_1$, P_1^{-1}, and P_2^{-1}

$A=$

$$
\begin{tikzcd}
\begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 5 \\ 3 & 4 & 9 \end{pmatrix} \arrow[r, "r_2 \to r_2 - 2r_1"] \arrow[d, "r_2 \leftrightarrow r_3", "P^{-1}"'] & \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 3 & 4 & 9 \end{pmatrix} \arrow[r, "r_3 \to r_3 - 3r_1"] & \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix} \arrow[d, "r_2 \leftrightarrow r_3"] \\
\begin{pmatrix} 1 & 1 & 2 \\ 3 & 4 & 9 \\ 2 & 2 & 5 \end{pmatrix} \arrow[r, "r_2 \to r_2 - 3r_1"] & \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 2 & 2 & 5 \end{pmatrix} \arrow[r, "r_3 \to r_3 - 2r_1"] & \underset{U}{\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}}
\end{tikzcd}
$$

$$A = PLU$$

$P =$ permutation

$L =$ lower triangular

$U =$ upper triangular.

$$L^{-1}P^{-1}A = U$$

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 4 & 9 \\ 2 & 2 & 5 \end{pmatrix}$$

$$PLU = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 5 \\ 3 & 4 & 9 \end{pmatrix}$$

Thm Every (square) matrix can be factorized as $A = PLU$, where

$P =$ permutation

$L =$ Lower triangular w/ 1's on diagonal

$U =$ REF

If A is invertible, U is upper triangular w/ non-zero entries on diagonal.

If A is not invertible, U has free variables, + rows of 0's at bottom.

A is $m \times n$ matrix $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$

the <u>transpose</u> of A, is an $n \times m$ matrix

$$A^T = \begin{pmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & & \\ a_{1n} & \cdots & a_{mn} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Columns of $A^T \approx$ rows of A
rows of $A^T \approx$ cols of A

$$(A^T)_{ij} = A_{ji}$$

Basic properties.

1) $(A+B)^T = A^T + B^T$

2) $(cA)^T = cA^T$

3) $(AB)^T = B^T A^T$ ($\underline{\text{NOT}}$ $A^T B^T$)

4) $(A^T)^{-1} = (A^{-1})^T$

$\underline{\text{Pf}}$| 1), 2) obvious

$$((AB)^T)_{ij} = (AB)_{ji} = \sum_k A_{jk} B_{ki}$$
$$= \sum_k A^T_{kj} B^T_{ik}$$
$$= \sum_k B^T_{ik} A^T_{kj} = (B^T A^T)_{ij}$$

If A is 3×4 and B is 4×5, AB is 3×5

A^T is 4×3 B^T is 5×4 $A^T B^T$ is DNE

$B^T A^T = 5\times3$

$I = AA^{-1}$, so $I = I^T = \underline{(A^{-1})^T A^T}$, so $(A^{-1})^T$ is inverse to A^T

$I = A^{-1}A$ so $I = I^T = A^T (A^{-1})^T$

$$x \cdot y = \sum x_i y_i = x^T y$$

Inner product

$$= (x_1, \dots \; x_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$y \cdot x = y^T x = (x^T y)^T = (x \cdot y)^T = (x \cdot y)$$

Outer product of x and y is $x y^T$

$$\underset{n \times 1}{\begin{pmatrix} x \end{pmatrix}} \underset{1 \times n}{\begin{pmatrix} \quad y^T \quad \end{pmatrix}} = \begin{pmatrix} n \times n \end{pmatrix}$$

A square matrix A is symmetric if $A^T = A$

Ex. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$, NOT $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Symmetric matrices $\Longleftrightarrow$ quadratic functions.

$$Q(x) = x^T A x$$

E.g. in $\mathbb{R}^2$, if $A = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$

$$Q\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1 \ x_2) \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= (x_1 \ x_2) \begin{pmatrix} ax_1 + bx_2/2 \\ bx_1/2 + cx_2 \end{pmatrix} = ax_1^2 + bx_1x_2 + cx_2^2$$

$$Q(x) = \sum_{ij} A_{ij} x_i x_j$$

$$A_{ij} = \frac{1}{2} \frac{\partial^2 Q}{\partial x_i \partial x_j}$$

In relativity

$$X \cdot Y = X_1 Y_1 + X_2 Y_2 + X_3 Y_3 - c^2 X_4 Y_4$$

$$= X^T \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & -c^2 \end{pmatrix} Y$$

If G is a symmetric matrix, can use generalized inner product

$$\langle x | y \rangle = x^T G y$$

If R is any matrix, $A = R^T R$ is symmetric.

pf $A^T = (R^T R)^T = R^T R^{TT} = R^T R = A$

$$\langle \cancel{A} x | y \rangle = x^T A y = x^T R^T R y = (Rx)^T (Ry)$$

$$= (Rx) \cdot (Ry)$$