Review

A vector in $\mathbb{R}^n$ is a list of n #s

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Can add component-wise
Can multiply by scalars.

Given vectors $\vec{v}_1, \dots \vec{v}_k$, can take
<u>linear combinations</u> $a_1\vec{v}_1 + \dots + a_k\vec{v}_k$

Given $\vec{b}$ in $\mathbb{R}^n$, 2 natural questions:

1) Is $\vec{b}$ a lin comb of $\{\vec{v}_1 \dots \vec{v}_k\}$?
2) If so, in how many ways?

Span $\{\vec{v}_1, \dots \vec{v}_k\}$ = {all possible linear combinations}.

$\{\vec{v}_1, \dots \vec{v}_k\}$ is linearly independent

if the only way to write

$\vec{0} = a_1\vec{v}_1 + \cdots + a_k\vec{v}_k$ is with $a_1 = a_2 = \cdots = a_k = 0$

~~The~~ Linearly dependent $\Leftrightarrow$ not independent.

Dependent $\Leftrightarrow$ you can write one as a lin. comb. of the others

$\Leftrightarrow$ you can write one as a lin. comb of the previous ones.

Ex

If $3\vec{v}_1 + 0\vec{v}_2 + 5\vec{v}_3 + 17\vec{v}_4 + 0\vec{v}_5 = 0$

$17\vec{v}_4 = -3\vec{v}_1 - 5\vec{v}_3$

$\vec{v}_4 = \frac{-3}{17}\vec{v}_1 - \frac{5}{17}\vec{v}_3$

A matrix is

1) An array of #s. (obvious, but useless)

2) Collection of ~~cat~~ columns

3) Collection of rows

4) Machine that acts on vectors.

5) Machine that acts on matrices.

A is $m \times n$ matrix

$$A = \left(\vec{a}_1 \cdots \vec{a}_n \right) \qquad \text{each } \vec{a}_i \in \mathbb{R}^m$$

Def: ~~$A\vec{x}$~~ If $\vec{x} \in \mathbb{R}^n$, $A\vec{x} = x_1 \vec{a}_1 + \cdots + x_n \vec{a}_n$

= linear combination of columns.

$$(Ax)_i = \sum_j A_{ij} x_j$$

$$\begin{cases} \text{Is } \vec{b} \text{ a linear combination of } \vec{a}_1, \dots \vec{a}_n? \\ \text{If so, in how many ways?} \end{cases}$$

= Find all solutions to $A\vec{x} = \vec{b}$

Column picture.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \vec{x} = \begin{pmatrix} 5 \\ 7 \end{pmatrix} \iff x_1 \underbrace{\begin{pmatrix} 1 \\ 4 \end{pmatrix}}_{\vec{a}_1} + x_2 \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{\vec{a}_2} + x_3 \underbrace{\begin{pmatrix} 3 \\ 6 \end{pmatrix}}_{\vec{a}_3} = \underbrace{\begin{pmatrix} 5 \\ 7 \end{pmatrix}}_{\vec{b}}$$

Row picture.

Each row is an equation.

$$x_1 + 2x_2 + 3x_3 = 5$$
$$4x_1 + 5x_2 + 6x_3 = 7$$

$(A \mid b)$

Manipulating equations = manipulating rows

= row-reduction

= Gauss elimination

or

Gauss-Jordan elimination.

3 basic row operations

1) Subtract l_{ij} (row j) from row r_i
2) Swap rows
3) Rescale row i

Can convert A or $A|b$ to REF.

Row-echelon-form (REF)

1) Any rows of 0's are at the bottom
2) First non-zero entry of each row (pivot) is to right of previous pivot.

Typically can convert to REF using only type-1 moves

$$A = LU$$

Sometimes need swaps $A = PLU$

To solve $Ax = b$

1) Write $[A \mid b]$

2) Row-reduce to $[U \mid c]$ with U in REF

3) Check for contradictions.

4) If no contradictions, identify pivot variables and free variables.

$$\left(\begin{array}{cccc|c} 3 & 1 & 4 & 8 & 5 \\ 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

x_1, x_3, x_4 pivot variables
x_2 = free variable.

5) Let free variables = whatever you want.

6) Each row gives pivot variable in terms of subsequent variables

7) Solve one variable at a time, starting at end and working backwards.

(Back substitution)

$$\#\text{ of solutions} = \begin{cases} 0 & \text{if contradictions} \\ 1 & \text{if no contradictions, no free (pivot in each column)} \\ \infty & \text{if no contradictions, at least one free.} \end{cases}$$

If A is square and U is upper triangular (with non-zero on diagonal)

e.g. $U = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 4 & 5 & 6 \\ 0 & 0 & -1 & 7 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

No contradictions

No free variables

Procedure is

1) Write $[A|b]$
2) Reduce to $[U|c]$
3) Back-substitute.

If A is square 2 possibilities

I) U ~~has~~ is triangular.

a) There is a unique soln to $Ax = b$
b) Columns of A are lin. independent.
c) Span (columns) $= \mathbb{R}^n$
d) A is invertible.

II) U has $< n$ pivots

a) Infinitely many soln to $Ax = 0 \Rightarrow$ Columns are lin dependent
b) For some $\vec{b}$, contradictions, so Span (columns) $\neq \mathbb{R}^n$.
c) A is singular.

$m \times n$

Matrix = machine that converts vectors in $\mathbb{R}^n$ to vectors in $\mathbb{R}^m$.

$$\underbrace{A}_{m\times n}\underbrace{\vec{x}}_{n} = \underbrace{\vec{y}}_{m} \qquad A : \mathbb{R}^n \longrightarrow \mathbb{R}^m.$$

(AB) = "first do B, then do A".

$$(AB)\vec{x} = A(B\vec{x})$$

If A is $m \times n$ B must be $n \times p$

AB is $m \times p$ If $B = (\vec{b}_1 \cdots \vec{b}_p)$

$$AB = (A\vec{b}_1 \cdots A\vec{b}_p)$$

$$(AB)_{ij} = \sum_k A_{ik} B_{kj} = \begin{pmatrix} \text{ith row of } A \end{pmatrix} \begin{pmatrix} \text{jth} \\ \text{col} \\ \text{of} \\ B \end{pmatrix}$$

A has an inverse A^{-1} if

$$AA^{-1} = A^{-1}A = I$$

Only square matrices have inverses.

If A^{-1} exists

$A(A^{-1}b) = \cancel{A} I b = b$, so $\vec{x} = A^{-1}b$ is a soln to $A\vec{x} = \vec{b}$

$x = A^{-1}(Ax) = A^{-1}b$, so $A^{-1}b$ is the only soln.

<u>How to find A^{-1}.</u>

$$(A \mid I) \to (U \mid \cancel{D} M_0) \to (I \mid \cancel{A^{-1}})$$

$$\cancel{M = A^{-1}}$$

$$(A \mid I) \to (I \mid A^{-1})$$

For 2×2 $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

If A is square $\quad A : \mathbb{R}^n \circlearrowleft$

$A^n =$ (apply A n times)

$A^{-n} =$ (apply A^{-1} n times) $= (A^{-1})^n$

$$(AB)^{-1} = B^{-1}A^{-1}$$

Matrices acting on matrices.

Each row operation is multiplication by a matrix.

To get that matrix, apply the operation to I.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \qquad \text{Subtracts } 3(r_2) \text{ from } r_3$$

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{array}{l} \text{multiples } r_1 \text{ by } 5 \\ \phantom{\text{multiples }} r_2 \text{ by } -1 \end{array}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{permutes}$$

$$\begin{tikzcd} r_1 \arrow[r] & r_2 \arrow[dl] \\ r_3 \arrow[u] & \end{tikzcd}$$

If $A \to U$, then
w/o swaps

$A = LU$, where

$$L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}$$

If swaps are required, $A = PLU$

$$x \cdot y = x^T y = |x||y| \cos\theta$$

$$|x| = \sqrt{x \cdot x}$$

$$(A^T)_{ij} = A_{ji} \qquad (AB)^T = B^T A^T$$