Defi $\vec{v} \cdot \vec{w} = \vec{v}^T \vec{w}$

We say $\vec{v}$ and $\vec{w}$ are orthogonal if $v^T w = 0$.

<u>Claim</u>: Every vector in $Nul(A)$ is $\perp$ to every row of A.

<u>Pf</u> If $A = \begin{pmatrix} r_1^T \\ \vdots \\ r_m^T \end{pmatrix}$

$$0 = Ax = \begin{pmatrix} r_1^T \\ \vdots \\ r_m^T \end{pmatrix} \begin{pmatrix} x \end{pmatrix} = \begin{pmatrix} r_1^T x \\ r_2^T x \\ \vdots \\ r_m^T x \end{pmatrix}$$

<u>Cor</u> Every vector is $Nul(A)$ is $\perp$ to every vector in $Row(A)$.

Claim: If $x \perp$ every ~~row~~ vector in $Row(A)$, then $x \in Nul(A)$.

<u>Pf</u> $Ax = \begin{pmatrix} r_1^T x \\ \vdots \\ r_m^T x \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ ✓

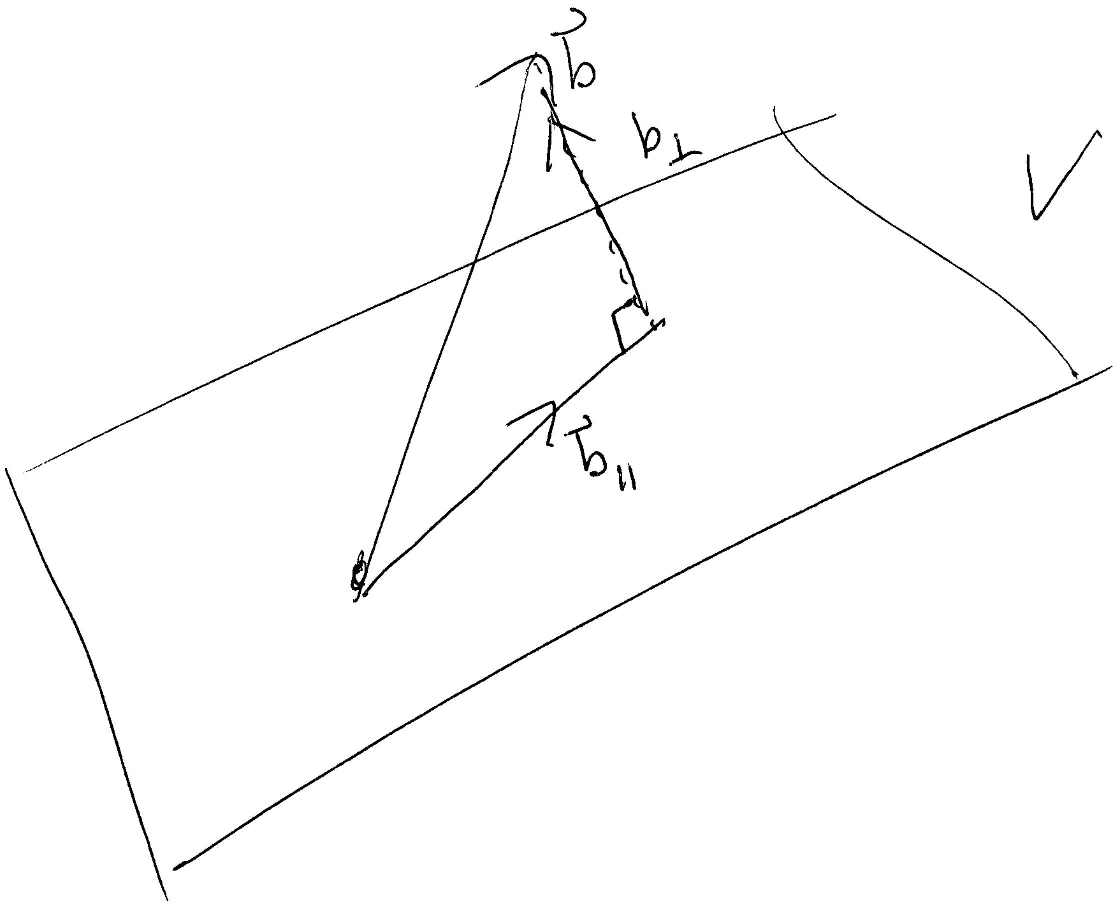
Let $V \subset \mathbb{R}^n$ be a subspace.

Def $V^\perp = \{\vec{x} \mid \vec{v} \cdot \vec{x} = 0 \text{ for every } \vec{v} \in V\}$

$= $ Orthogonal complement to V.

$$\left.\begin{aligned} \mathrm{Null}(A) &= (\mathrm{Row}(A))^\perp \\ \mathrm{Row}(A) &= (\mathrm{Nul}(A))^\perp \end{aligned}\right\} \text{ Subspaces of } \mathbb{R}^n$$

$$\left.\begin{aligned} \mathrm{Nul}(A^T) &= (\mathrm{Col}(A))^\perp \\ \mathrm{Col}(A) &= \mathrm{Nul}(A^T)^\perp \end{aligned}\right\} \text{ Subspaces of } \mathbb{R}^m.$$



Every vector $\vec{b}$ can be written as

$\vec{b} = \vec{b}_\parallel + \vec{b}_\perp$, where $\vec{b}_\parallel \in V$
$\vec{b}_\perp \in V^\perp$

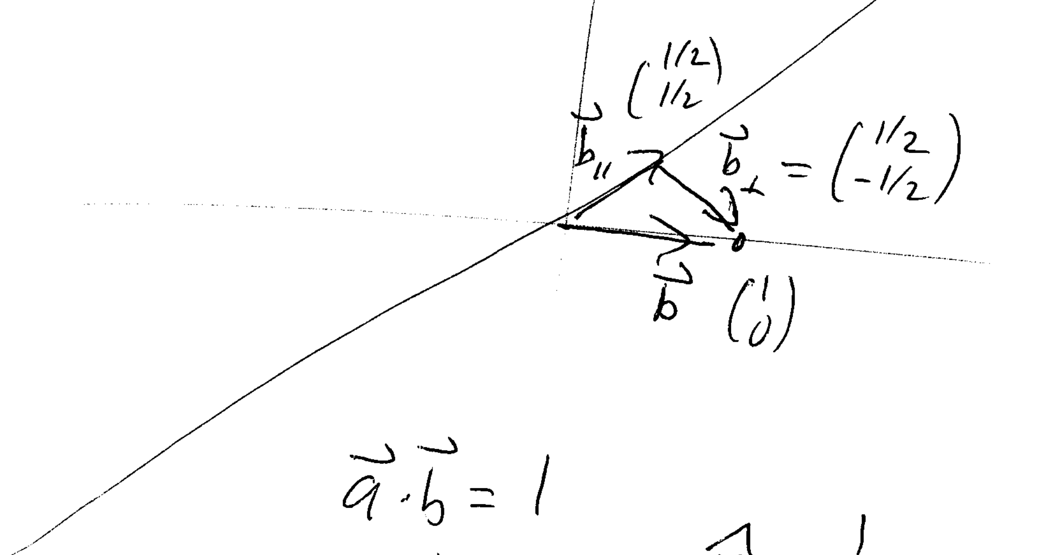
$\vec{b}_\parallel$ is point in V closest to $\vec{b}$,

$|\vec{b}_\perp|$ is distance from $\vec{b}$ to V.

Ex: In $\mathbb{R}^2$, $\vec{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\vec{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

V



$\vec{a} \cdot \vec{b} = 1$

$\vec{a} \cdot \vec{a} = 2$ $\qquad \hat{x} = \frac{1}{2}$

$\vec{b}_{\perp} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$ $\qquad \vec{b}_{\parallel} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$

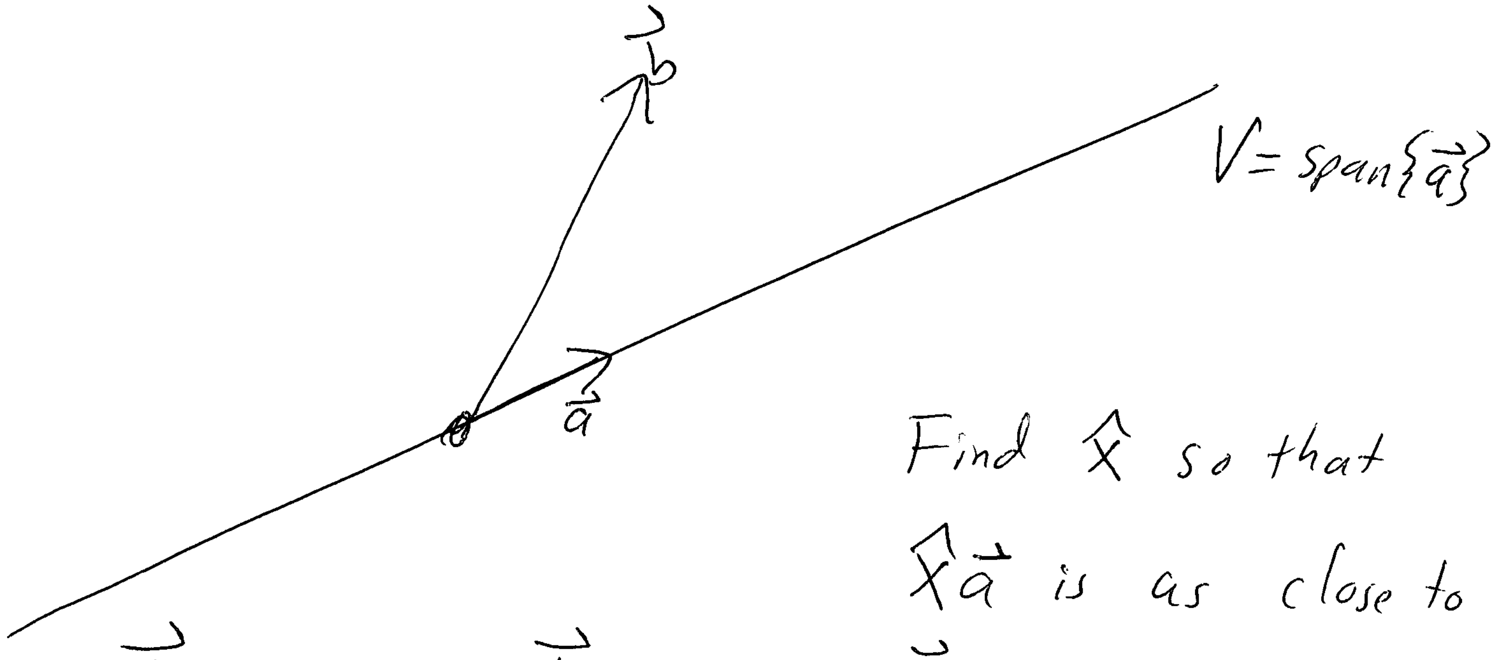
Ex: In $\mathbb{R}^3$, $\vec{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$\vec{a} \cdot \vec{b} = 6$

$\vec{b} \cdot \vec{b} = 14$

$\vec{b}_{\parallel} = \frac{6}{14} \vec{a}$ $\qquad \hat{x} = \frac{6}{14}$

$\vec{b}_{\perp} = \vec{b} - \frac{3}{7} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$



Find $\hat{x}$ so that $\hat{x}\vec{a}$ is as close to $\vec{b}$ as possible.

$$\vec{b} = \underbrace{\hat{x}\vec{a}}_{\vec{b}_{\parallel}} + \vec{b}_{\perp}$$

$$\vec{b}_{\perp} = \vec{b} - \hat{x}\vec{a}$$

$$0 = \vec{a}\cdot\vec{b}_{\perp} = \vec{a}\cdot\vec{b} - \hat{x}\,\vec{a}\cdot\vec{a} \quad ;$$

Scalar projection of $\vec{b}$ onto $\vec{a}$

$$\boxed{\hat{x} = \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}} = \frac{a^T b}{a^T a}}$$

$$\boxed{\vec{b}_{\parallel} = \vec{a}\,\frac{(a^T b)}{a^T a}}$$

Vector projection.

$$\vec{b}_{\parallel} = \vec{a}\,\frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}} = \left(\frac{a\,a^T}{a^Ta}\right) b$$

Ex1: $\vec{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad a^T = \begin{pmatrix} 1 & 1 \end{pmatrix}$

$$a\,a^T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad a^Ta = 2$$

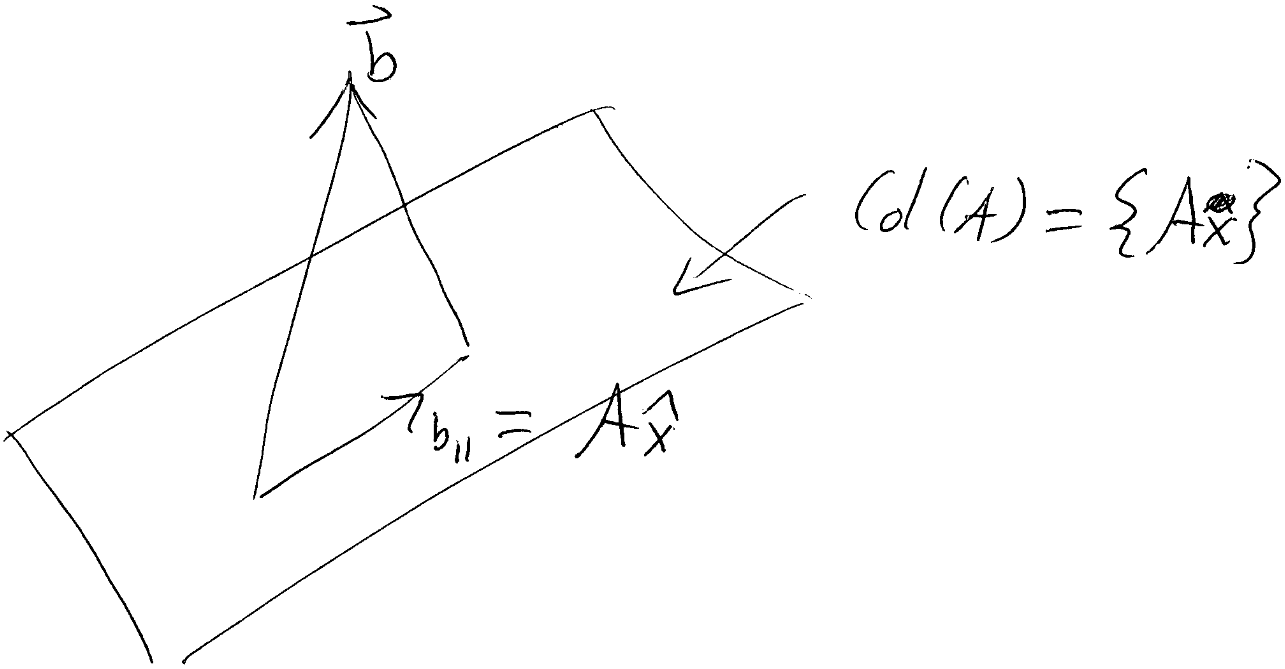
$$P_{\vec{a}} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$P_{\vec{a}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

$$P_{\vec{a}} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$$\begin{array}{ccccc} \vec{b} & & \vec{b}_{\parallel} & & \vec{b}_{\perp} \\ \begin{pmatrix} 3 \\ 5 \end{pmatrix} & = & \begin{pmatrix} 4 \\ 4 \end{pmatrix} & + & \begin{pmatrix} -1 \\ 1 \end{pmatrix} \end{array}$$

Projection onto $Col(A) = span\{\vec{a}_1 \cdots \vec{a}_n\}$.



Find $\hat{x}$ s.t. $A\hat{x}$ is as close as possible to $\vec{b}$. $(A\hat{x} = \vec{b}_{\parallel})$

$$\vec{b}_{\perp} = \vec{b} - A\hat{x} \text{ is in } (Col(A))^{\perp} = Nul(A^T)$$

$$0 = A^T\vec{b}_{\perp} = A^Tb - A^TA\hat{x}$$

Solve $A^TA\hat{x} = A^Tb$ if A^TA invertible

$$\hat{x} = (A^TA)^{-1}A^Tb$$

$$\vec{b}_{\parallel} = P_V b = Ax = A(A^TA)^{-1}A^T b$$

$$P_V = A(A^TA)^{-1}A^T$$

<u>Thm</u> A^TA is invertible if and only if columns of A are linearly independent (That is, if only solution to $Ax=0$ is $x=0$).

Claim: $Nul(A^TA) = Nul(A)$.

$\Rightarrow$ If $Ax=0$ $\quad A^TAx = A^T0 = 0$.

$\Leftarrow$ If $A^TAx=0$, $\quad x^TA^TAx = 0$

$$(Ax)^TAx = 0$$

$$(Ax)\cdot(Ax) = 0$$

$$Ax = 0$$

If ~~the~~ Columns of A are lin ind, then

1) A^TA is invertible

2) There is a <u>unique</u> soln to

$$A^TA\hat{x} = A^T\vec{b}.$$

3) There is a unique $\hat{x}$ s.t.

$$A\hat{x} = \vec{b}_{\parallel}$$

If columns aren't lin ind, then $\vec{b}_{\parallel}$ is still well-defined, but there are many $\hat{x}$ with $A^TA\hat{x} = A^T\vec{b}$.

Projection onto Col(A)

$$= \text{Projection onto Col} \underbrace{\begin{pmatrix}1&2\\2&5\\3&7\end{pmatrix}}_{\tilde{A}}$$

$$\tilde{A}^T\tilde{A} = \begin{pmatrix}1&2&3\\2&5&7\end{pmatrix}\begin{pmatrix}1&2\\2&5\\3&7\end{pmatrix} = \begin{pmatrix}14&33\\33&78\end{pmatrix}$$

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix}d&-b\\-c&a\end{pmatrix} = \frac{1}{\cancel{\text{nasty}}\ 3}\begin{pmatrix}78&-33\\-33&14\end{pmatrix}$$

$3 = ad-bc$

Δ

$$P_V = \frac{1}{3}\begin{pmatrix}1&2\\2&5\\3&7\end{pmatrix}\begin{pmatrix}78&-33\\-33&14\end{pmatrix}\begin{pmatrix}1&2&3\\2&5&7\end{pmatrix} = \begin{pmatrix}2/3&-1/3&1/3\\-1/3&2/3&1/3\\1/3&1/3&2/3\end{pmatrix}$$

$$A^TA\hat{x} = A^Tb$$