2nd Midterm Thursday

Same ground rules as 1st:

1) Closed book
2) No calculators
3) One $8\frac{1}{2} \times 11$ crib sheet OK.

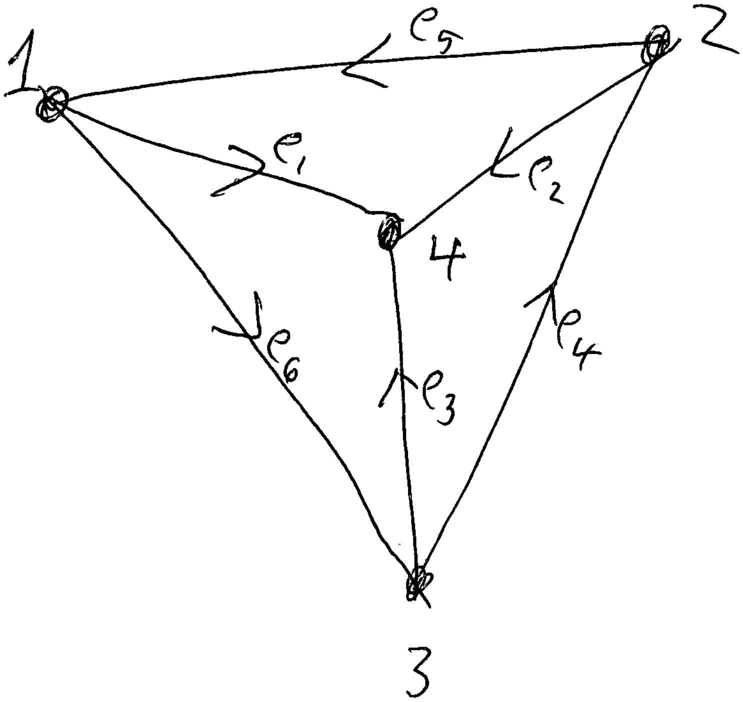
5 problems: 20 pts each

Read the directions!

Answer the whole question.

$\boxed{\text{Justify your answers.}}$

Electrical Circuits.



Nodes have voltages.

Edges have currents.

Kirchoff's Laws:

Voltage: $\cancel{\sum_{\text{loop}}}$ $\sum_{\substack{\text{edges in} \\ \text{a loop}}} (\text{voltage change}) = 0.$

Current: $\sum$ ~~a~~ currents going into a vertex $= 0$.

Incidence Matrix

$$A = \begin{array}{cc} & \text{nodes} \\ \text{edges} & \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

$$\vec{V} = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix} = \text{voltages}$$

$$A\vec{V} = \begin{pmatrix} V_4 - V_1 \\ V_4 - V_2 \\ V_4 - V_3 \\ V_2 - V_3 \\ V_1 - V_2 \\ V_3 - V_1 \end{pmatrix}$$

$$A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

$\text{Dim Nul}(A) = 1$

$\text{Rank}(A) = (\#\text{vertices}) - 1$

$$A^T = \begin{pmatrix} -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\vec{i} = \begin{pmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{pmatrix}$$

current

$$A^T \vec{i} = \begin{pmatrix} i_5 - i_1 - i_6 \\ i_4 - i_2 - i_5 \\ i_6 - i_3 - i_4 \\ i_1 + i_2 + i_3 \end{pmatrix} = 0 \quad \text{by KCL}$$

$$\vec{i} \in \mathrm{Nul}(A^T)$$

$$\begin{aligned} \dim \mathrm{Nul}(A^T) &= \mathrm{width}(A^T) - \mathrm{rank}(A^T) \\ &= (\#\text{edges}) - (\#\text{vertices} - 1) \\ &= 1 + \#\text{edges} - \#\text{vertices} \end{aligned}$$

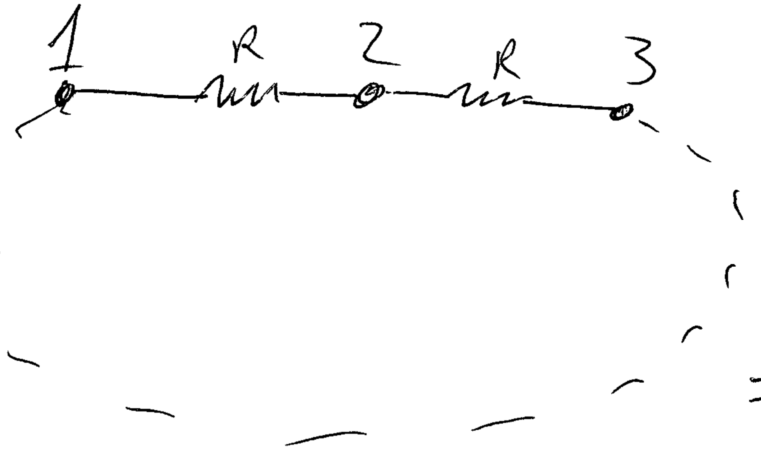
$$C = \begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_6 \end{pmatrix} = \begin{pmatrix} \frac{1}{r_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{r_6} \end{pmatrix}$$

$\cancel{I}$ $i_k = c_k \cdot$ (voltage difference at edge k).

$$\vec{i} = C \cdot A \cdot \begin{pmatrix} \text{voltages} \end{pmatrix}$$

$$0 = A^T \vec{i} = (A^T C A) \cdot (\text{voltage})$$

$$\text{voltage} \in \mathrm{Nul}(A^T C A)$$



$$X_2 = \frac{X_1 + X_3}{2}$$

Current into node 2.

$$= (X_2 - X_1) \cdot \frac{1}{R} + (X_2 - X_3) \cdot \frac{1}{R}$$

$$= 0 = (2X_2 - X_1 - X_3) \cdot \frac{1}{R}$$

Vector space = Set where you can sensibly take linear combinations.

Subspace = Subset W of V s.t.

1) W closed under addition
2) " " " scalar mult.
3) $0 \in W$

~~If~~ $A =$ $m \times n$ matrix, 4 fundamental Subspaces.

$\mathrm{Nul}(A) = \mathrm{Ker}(A), \mathrm{Row}(A) \subset \mathbb{R}^n$

$\mathrm{Col}(A), \mathrm{Nul}(A^T) \subset \mathbb{R}^m$.

$\mathrm{Nul}(A) = \{\text{all solutions to } A\vec{x}=0\}$.

$= \mathrm{Nul}(A_{rref}) = (\mathrm{Row}(A))^{\perp}$

Basis = special solutions

dimension = n − rank. = # free variables

~~Row (A)~~

$\mathrm{Col}(A) = \mathrm{Span}(\text{columns}) = \{Ax\} = (\mathrm{Nul}(A^T))^{\perp}$

$= \{\vec{b} \in \mathbb{R}^m \mid A\vec{x}=\vec{b} \text{ has a solution}\}$.

Basis = pivot columns of A

dimension = rank.

$\mathrm{Row}(A) = \mathrm{Col}(A^T) = \mathrm{Row}(A_{rref}) = (\mathrm{Nul}(A))^{\perp}$

Basis = nonzero rows of A_{rref}

dimension = rank.

$\mathrm{Nul}(A^T)$ = Solutions to $A^T x = 0$ = $(\mathrm{Col}(A))^{\perp}$

dimension = m − rank.

Rank + A_{rref} tell you everything.

To solve $\vec{A}x = \vec{b}$

1) Reduce $(A|\vec{b}) \Rightarrow (A_{rref} | \vec{d})$

2) If there are contradictions, stop $\Rightarrow$ no soln.

3) In no ~~soln~~ contradictions, find a particular soln $\vec{X}_p$. ($X_{free} = 0$, pivot variables = entries of $\vec{d}$)

4) Soln to $Ax=b$ = $\vec{X}_p$ + $Null(A)$

$$\underbrace{\begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{pmatrix}}_{A} \left| \underbrace{\begin{matrix} 1 \\ 6 \\ 7 \end{matrix}}_{b} \right. \Rightarrow \left(\begin{matrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{matrix} \left| \begin{matrix} 1 \\ 6 \\ 0 \end{matrix} \right.\right)$$

$$\begin{aligned} X_1 &= 1 - 3X_2 - 2X_4 \\ X_2 &= X_2 \\ X_3 &= 6 - 4X_4 \\ X_4 &= X_4 \end{aligned}$$

$$\vec{X} = \begin{pmatrix} 1 \\ 0 \\ 6 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 0 \\ -4 \\ 1 \end{pmatrix}$$

A set of vectors $\vec{v}_1, \dots \vec{v}_n$ is linearly independent if the only way to write

$$0 = c_1\vec{v}_1 + \dots + c_n\vec{v}_n \text{ is with } c_1 = c_2 = \dots = 0.$$

These vectors span V if

$$\{\text{linear combinations}\} = V$$

If $V = \mathbb{R}^m$

- If $m < n$ too many vectors, not lin ind.
- If $m > n$ not enough vectors, don't span.
- If $m = n$: both or neither property.

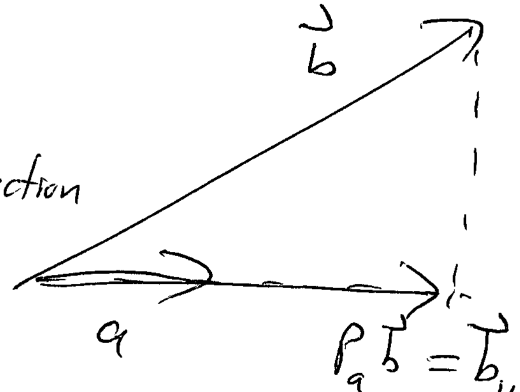
Look at matrix $A = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix}$

Linear independence $\Leftrightarrow$ pivot in each column

Span $\Leftrightarrow$ pivot in each row.

Projection onto a vector $\vec{a}$.

$$P_{\vec a}\vec b = \left(\frac{\vec a\cdot\vec b}{\vec a\cdot\vec a}\right)\vec a = \hat x\,\vec a = \text{vector projection.}$$



$$\hat x = \frac{\vec a\cdot\vec b}{\vec a\cdot\vec a} = \text{scalar projection}$$

$$\vec b_\perp = \vec b - P_{\vec a}\vec b = \vec b - A\hat x = \text{error}$$

$$\vec b_\perp\cdot\vec a = 0$$

$$V = Col(A)$$

$$P_A\vec b = \boxed{A\hat{\vec x}}\ , \quad \text{where } \boxed{A^TA\hat{\vec x} = A^T\vec b}$$

Call this a least-squares soln to $A\vec x = \vec b$.

If cols of A are lin ind, A^TA is invertible,

$$\hat x = (A^TA)^{-1}A^T\vec b$$

$$P_A = A(A^TA)^{-1}A^T \quad \text{(ugly)}$$

If cols of A are $\perp$, then

$$P_A = P_{\vec{a}_1} + P_{\vec{a}_2} + \cdots + P_{\vec{a}_n}$$

Gram-Schmidt process.

Given: vectors $\vec{x}_1, \vec{x}_2, \ldots$ (linearly independent)

Goal vectors $\vec{y}_1, \vec{y}_2 \ldots$ orthogonal

with $\text{Span}(\vec{y}_1) = \text{Span}(\vec{x}_1)$

$\text{Span}(\vec{y}_1, \vec{y}_2) = \text{Span}(\vec{x}_1, \vec{x}_2)$

$\vdots$

$$\vec{y}_1 = \vec{x}_1$$

$$\vec{y}_2 = \vec{x}_2 - P_{\vec{y}_1} \vec{x}_2$$

$$\vec{y}_3 = \vec{x}_3 - P_{\vec{y}_1} \vec{x}_3 - P_{\vec{y}_2} \vec{x}_3$$

$\vdots$

$\vec{y}_k = \vec{x}_k -$ (projection of $\vec{x}_k$ onto previous $\vec{y}$'s)

$\neq \vec{x}_k -$ (projection of $\vec{x}_k$ onto previous $\vec{x}$'s)