Determinants:

$\mathrm{Det}(A) =$ ugly formula with $n!$ terms.

What's it good for:

$\mathrm{Det}(A) = \pm$ Volume of parallelogram spanned by columns of A

$=$ same with rows

$\mathrm{Det} = 0 \iff$ vectors are linearly dependent

$\iff A$ is singular.

$\mathrm{Det} \neq 0 \iff$ vectors are linearly independent

$\iff$ vectors span $\mathbb{R}^n$

$\iff A$ is invertible.

In $\mathbb{R}^1$. $\quad A = (a)$

length of (a) is $\quad a \quad$ if $a > 0$

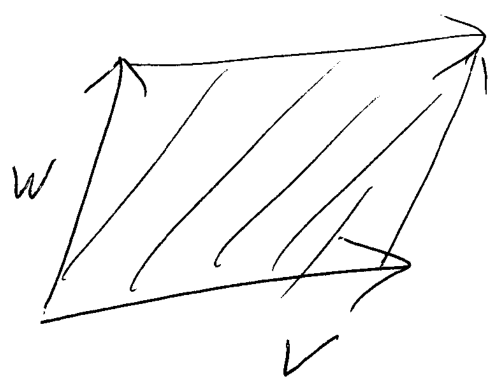
$\qquad\qquad\qquad\quad -a \quad$ if $a < 0$

length $= |a|$

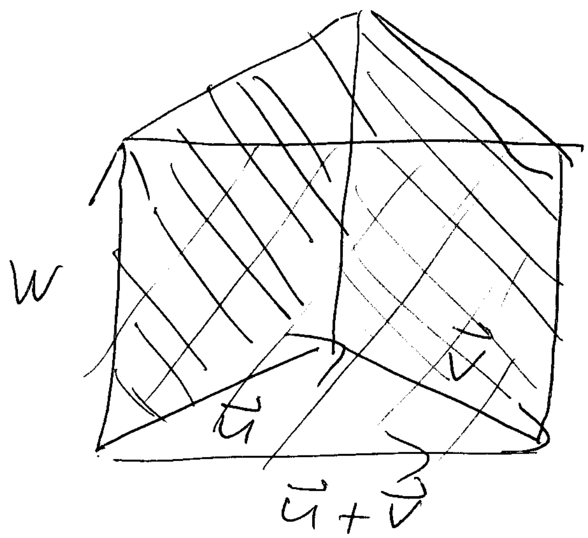
$a = \pm$ length.

$\vec{v}, \vec{w} \in \mathbb{R}^2$.

Area $(\vec{v}, \vec{w}) =$ area of



$$\text{Area}(\vec{u}+\vec{v},\vec{w}) = A(\vec{u}, w) + A(\vec{v},\vec{w})$$



$$\text{Area}(\vec{u},\vec{v}+\vec{w}) = A(\vec{u},\vec{v}) + A(\vec{u},\vec{w})$$

$$\text{Area}(c\vec{v},\vec{w}) = c\,A(\vec{v},\vec{w}) = A(\vec{v}, c\vec{w})$$

$$A(\vec{v},\vec{v}) = 0 = A(\vec{w},\vec{w}) = A(\vec{v}+\vec{w},\vec{v}+\vec{w})$$

$$0 = A(\vec{v}+\vec{w},\vec{v}+\vec{w}) = A(\vec{v},\vec{w}) + \cancel{A(\vec{v},\vec{v})} + A(\vec{w},\vec{v}) + \cancel{A(\vec{w},\vec{w})}$$

$$A(\vec{w},\vec{v}) = -A(\vec{v},\vec{w})$$

$$\vec{v} = (a,b) = a\vec{i} + b\vec{j}$$

$$\vec{w} = (c,d) = c\vec{i} + d\vec{j}$$

$$\text{Area}(\vec{v},\vec{w}) = \text{Area}(a\vec{i} + b\vec{j},\ c\vec{i} + d\vec{j})$$

$$= \cancel{\text{Area}}\ ac\,\cancelto{0}{\text{Area}(\vec{i},\vec{i})} + ad\,\text{Area}\cancelto{1}{(\vec{i},\vec{j})}$$

$$+ bc\,\text{Area}\cancelto{-1}{(\vec{j},\vec{i})} + bd\,\cancel{\text{Ad}}\,\text{Area}\cancelto{0}{(\vec{j},\vec{j})}$$

$$= ad - bc$$

$$\text{Det}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\text{Det}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \text{Det}\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$n = 3$

$Vol(\vec{u}, \vec{v}, \vec{w})$

1) Linear in $\vec{u}$
2) Linear in $\vec{v}$
3) Linear in $\vec{w}$.

4) If any two are same, $Vol = 0$
5) Swapping 2 vectors changes sign.

$$0 = Vol(\vec{u}+\vec{v}, \vec{u}+\vec{v}, \vec{w}) = \underbrace{\cancel{Vol(\vec{u}, \vec{u}, \vec{w})}}_{0} + Vol(\vec{u}, \vec{v}, \vec{w}) + Vol(\vec{v}, \vec{u}, \vec{w}) + \underbrace{\cancel{Vol(\vec{v}, \vec{v}, \vec{w})}}_{0}$$

$$Vol(\vec{v}, \vec{u}, \vec{w}) = -Vol(\vec{u}, \vec{v}, \vec{w})$$

6) $Vol(\vec{i}, \vec{j}, \vec{k}) = Vol(\vec{j}, \vec{k}, \vec{i}) = Vol(\vec{k}, \vec{i}, \vec{j}) = 1$

$Vol(\vec{i}, \vec{k}, \vec{j}) = Vol(\vec{k}, \vec{j}, \vec{i}) = Vol(\vec{j}, \vec{i}, k) = -1$

$\vec{u} = a\vec{i} + b\vec{j} + c\vec{k}$

$\vec{v} = d\vec{i} + e\vec{j} + f\vec{k}$

$\vec{w} = g\vec{i} + h\vec{j} + l\vec{k}$

$\mathrm{Vol}(\vec{u},\vec{v},\vec{w}) = \Sigma$ 27 terms.

21 terms are zero. 6 are left.

$$ael\,\mathrm{Vol}(\vec{i},\vec{j},\vec{k}) + bfg\,\mathrm{Vol}(j,k,i) + cdh\,\mathrm{Vol}(k,i,j)$$

$$+ afh\,\mathrm{Vol}(i,k,j) + bdl\,\mathrm{Vol}(j,i,k) + ceg\,\mathrm{Vol}(k,j,i)$$

$$= \left[ael + bfg + cdh - afh - bdl - ceg.\right]$$

$$\det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 7 \end{pmatrix} = \begin{array}{ccccc} + & + & + & - & - \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 & 2 \\ 1 & 3 & 7 & 1 & 3 \end{array}\ -$$

$$= 14 + 3 + 3 - 2 - 9 - 7$$

$$= 2$$

Examples:

$$\det\begin{pmatrix}1&1&1\\1&2&3\\1&3&7\end{pmatrix} \to \text{~~row redu~~}$$

$$= \det\begin{pmatrix}1&1&1\\0&1&2\\0&2&6\end{pmatrix} = \det\begin{pmatrix}1&1&1\\0&1&2\\0&0&2\end{pmatrix} = 2$$

also:

$$\det\begin{pmatrix}1&1&1\\1&2&3\\1&3&7\end{pmatrix} = \det\begin{pmatrix}1&0&0\\1&2&3\\1&3&7\end{pmatrix}$$

$$+\det\begin{pmatrix}0&1&0\\1&2&3\\1&3&7\end{pmatrix} + \det\begin{pmatrix}0&0&1\\1&2&3\\1&3&7\end{pmatrix}$$

Aim for cofactor formula.

$$\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \qquad \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \begin{array}{l} - \\ \\ + \end{array}$$

$-$

$$\det \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$+$

$$\boxed{\det(A) = \sum_{\substack{\text{all permutations} \\ \sigma \text{ of } (1,2,\dots n)}} \pm\, a_{1\sigma_1} a_{2\sigma_2} \cdots a_{n\sigma_n}}$$

def.

$+$ for even permutations (# flips = even)

$-$ for odd permutations (# flips = odd)

3 Row operations:

1) Add a multiple of one row to another
2) Swap 2 rows
3) Rescale a row.

What happens to determinant?

1) $\det\begin{vmatrix} \vec{v} \\ \vec{w} \\ \vec{x} \end{vmatrix}$ vs $\det\begin{vmatrix} \vec{v} \\ \vec{w}+3\vec{v} \\ \vec{x} \end{vmatrix}$

$$\det\begin{vmatrix} \vec{v} \\ \vec{w}+3\vec{v} \\ \vec{x} \end{vmatrix} = \det\begin{vmatrix} \vec{v} \\ \vec{w} \\ \vec{x} \end{vmatrix} + \det\begin{vmatrix} \vec{v} \\ 3\vec{v} \\ \vec{x} \end{vmatrix}$$

$$= \det\begin{vmatrix} \vec{v} \\ \vec{w} \\ \vec{x} \end{vmatrix}$$

Operation 1 doesn't change det!

Operation 2 flips sign of det.

Operation 3 rescales det.

$$\det \begin{pmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{pmatrix} = \text{product of diagonal entries.}$$

$$\det \begin{pmatrix} a_{11} & & (\text{stuff}) \\ & a_{22} & \\ 0 & & \ddots \\ & & & a_{nn} \end{pmatrix} = \text{same.}$$

$$\det \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ \text{stuff} & & a_{nn} \end{pmatrix} = \text{same.}$$

$\det A = \pm \prod$ pivots in REF obtained without rescaling.

$$\det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 7 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 6 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} = 2$$

$$\det(A^T) = \det(A)$$

$$\det(AB) = \det(A)\,\det(B)$$

$$\det(A)\,\det(A^{-1}) = \det(AA^{-1}) = \det(I) = 1$$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$$\mathbb{R}^n \xrightarrow{\quad A \quad} \mathbb{R}^n$$

Volume of output $= \det(A)$ (volume of input)

$\vec{e}_1, \dots \vec{e}_n \longrightarrow (A\vec{e}_1, \dots A\vec{e}_n) =$ columns of A

$=$ rows of A^T

volume $1 \longrightarrow \qquad \det(A^T) = \det(A)$

$$\det \begin{vmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 1 \\ 1 & 4 & 1 & 4 \\ 2 & 7 & 1 & 8 \end{vmatrix}$$

$$= \det \begin{vmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 4 & 1 \\ 1 & 4 & 1 & 4 \\ 2 & 7 & 1 & 8 \end{vmatrix} + \det \begin{vmatrix} 0 & 2 & 0 & 0 \\ 3 & 1 & 4 & 1 \\ 1 & 4 & 1 & 4 \\ 2 & 7 & 1 & 8 \end{vmatrix}$$

$$+ \det \begin{vmatrix} 0 & 0 & 3 & 0 \\ 3 & 1 & 4 & 1 \\ 1 & 4 & 1 & 4 \\ 2 & 7 & 1 & 8 \end{vmatrix} + \det \begin{vmatrix} 0 & 0 & 0 & 4 \\ 3 & 1 & 4 & 1 \\ 1 & 4 & 1 & 4 \\ 2 & 7 & 1 & 8 \end{vmatrix}$$

$$= 1 \det \begin{vmatrix} 1 & 4 & 1 \\ 4 & 1 & 4 \\ 7 & 1 & 8 \end{vmatrix} - 2 \det \begin{vmatrix} 3 & 4 & 1 \\ 1 & 1 & 4 \\ 2 & 1 & 8 \end{vmatrix} + 3 \det \begin{vmatrix} 3 & 1 & 1 \\ 1 & 4 & 4 \\ 2 & 7 & 8 \end{vmatrix}$$

$$- 4 \det \begin{vmatrix} 3 & 1 & 4 \\ 1 & 4 & 1 \\ 2 & 7 & 1 \end{vmatrix}$$