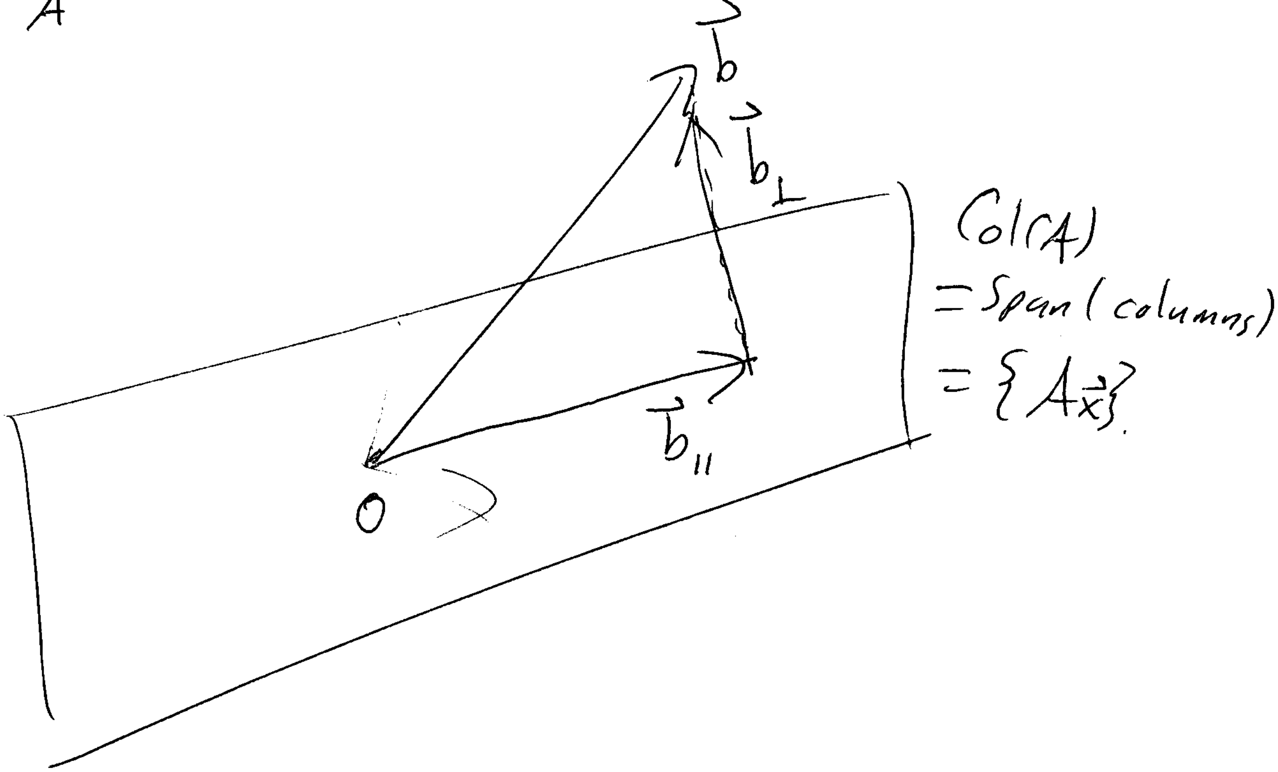
Matrix A



1) Find ~~x that ge~~ the point in $Col(A)$ that is closest to $\vec{b}$.
2) Find $\vec{x}$ s.t. $|\vec{b} - A\vec{x}|$ is minimized
3) Find $\vec{x}$ that minimizes $\sum \left(b_i - (Ax)_i\right)^2$
4) Solve $A\vec{x} = \vec{b}_\parallel = \vec{b} - \vec{b}_\perp$
5) Solve $(A^T A)\vec{x} = A^T\vec{b}$

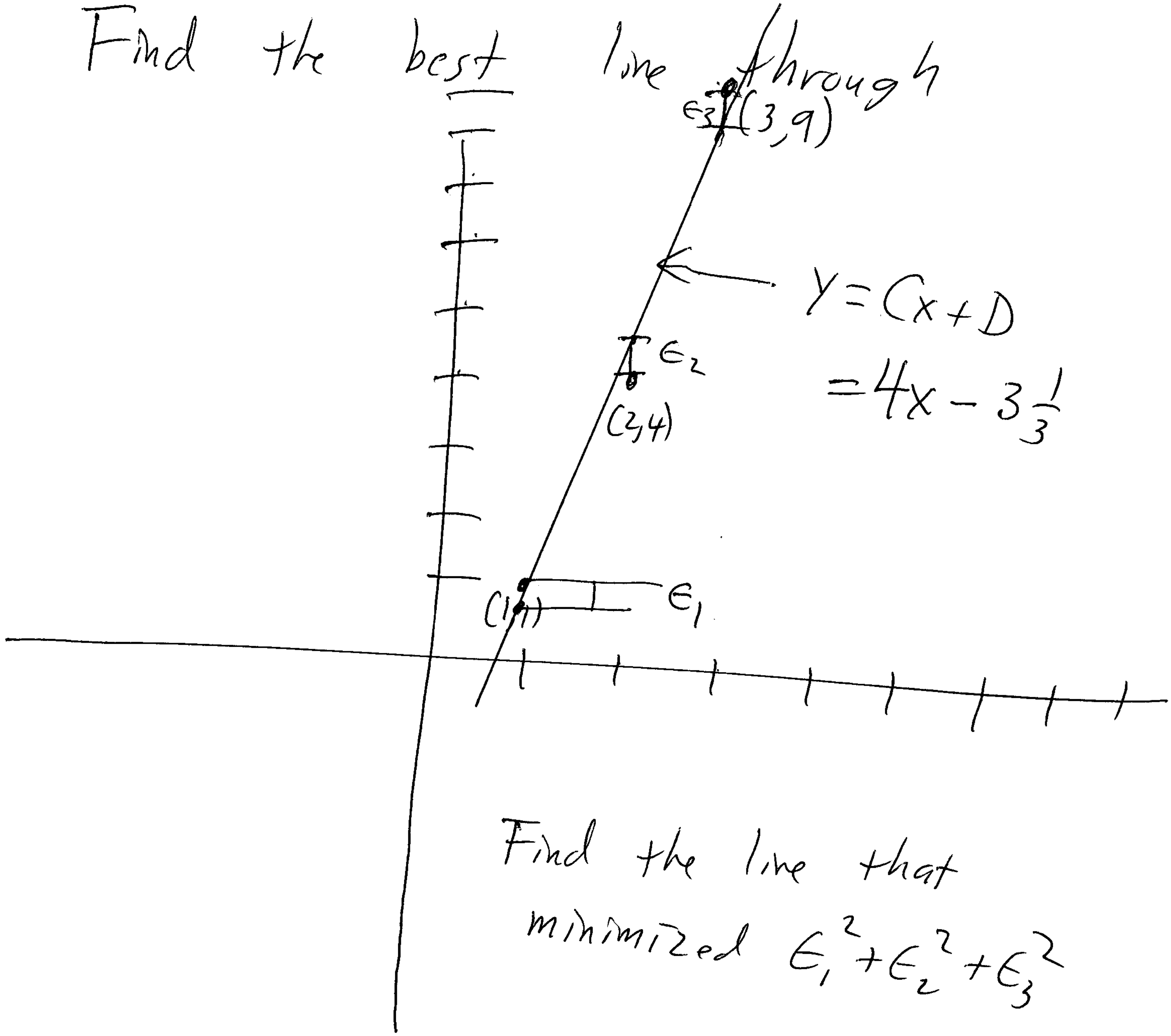
Same Question

Solution is denoted $\hat{x}$

Df: A least-squares solution to $A\vec{x}=\vec{b}$ is a value of $\vec{x}$ that minimized $|\vec{b}-A\vec{x}|$.

Thm A least-squares soln to $A\vec{x}=\vec{b}$ is an actual soln to $(A^TA)\vec{x}=A^T\vec{b}$.

Find the best line through



Find the line that minimized $\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2$

$$C\cdot 1 + D = 1 = \tfrac{2}{3} + \epsilon_1 \qquad \epsilon_1 = 1/3$$

$$C\cdot 2 + D = 4 = 4\tfrac{2}{3} - \tfrac{2}{3} \qquad \epsilon_2 = \tfrac{-2}{3}$$

$$C\cdot 3 + D = 9 = 8\tfrac{2}{3} + \tfrac{1}{3} \qquad \epsilon_1 = 1/3$$

$$\underset{A}{\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}} \underset{\vec{x}}{\begin{pmatrix} C \\ D \end{pmatrix}} = \underset{\vec{b}}{\begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix}}$$

$$A^T = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 14 & 6 \\ 6 & 3 \end{pmatrix} \qquad A^T b = \begin{pmatrix} 36 \\ 14 \end{pmatrix}$$

$$\begin{pmatrix} 14 & 6 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 36 \\ 14 \end{pmatrix}$$

$$\begin{pmatrix} C \\ D \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 3 & -6 \\ -6 & 14 \end{pmatrix} \begin{pmatrix} 36 \\ 14 \end{pmatrix} = \begin{pmatrix} 4 \\ -3\frac{1}{3} \end{pmatrix}$$

Best line through (x_1, y_1)

$$(x_2, y_2)$$

$$\vdots$$

$$(x_n, y_n)$$

$$y = Cx + D$$

$$\begin{aligned} Cx_1 + D &= y_1 \\ Cx_2 + D &= y_2 \\ &\vdots \\ Cx_n + D &= y_n \end{aligned} \qquad \begin{pmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$A^T = \begin{pmatrix} x_1 & \cdots & x_n \\ 1 & \cdots & 1 \end{pmatrix} \qquad A^T A = \begin{pmatrix} \sum x_i^2 & \sum x_i \\ \sum x_i & n \end{pmatrix}$$

$$A^T \vec{b} = \begin{pmatrix} \sum x_i y_i \\ \sum y_i \end{pmatrix} \qquad (A^T A)^{-1} = \frac{1}{n \sum x_i^2 - (\sum x_i)^2} \begin{pmatrix} n & -\sum x_i \\ -\sum x_i & \sum x_i^2 \end{pmatrix}$$

$$\begin{pmatrix} C \\ D \end{pmatrix} = (A^TA)^{-1}(A^Tb)$$

$$= \frac{1}{n\sum x_i^2 - (\sum x_i)^2} \begin{pmatrix} n & -\sum x_i \\ -\sum x_i & \sum x_i^2 \end{pmatrix} \begin{pmatrix} \sum x_i y_i \\ \sum y_i \end{pmatrix}$$

$$C = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - (\sum x_i)^2} = \text{slope}$$

$$D = \cancel{n\sum x_i^2} \frac{\sum x_i^2 \sum y_i - \sum x_i y_i \sum x_i}{n\sum x_i^2 - (\sum x_i)^2} = \text{intercept.}$$

If you want $y = C + Dx$.

$$A = \begin{pmatrix} 1 & x_1 \\ 1 & \vdots \\ 1 & \\ 1 & x_n \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & - & - & - & x_n \end{pmatrix}$$

$$A^TA = \begin{pmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{pmatrix} \qquad A^T\vec{b} = \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}$$

$$\text{slope} = \frac{\langle xy\rangle - \langle x\rangle\langle y\rangle}{\langle x^2\rangle - \langle x\rangle^2}$$

where $\langle\ \rangle$ means average

$$\text{intercept} = \frac{\langle x^2\rangle\langle y\rangle - \langle xy\rangle\langle x\rangle}{\langle x^2\rangle - \langle x\rangle^2}$$

How do you fit a parabola to data?

$$y = C + Dx + Ex^2$$

$$\begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 \end{pmatrix} \begin{pmatrix} C \\ D \\ E \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$A \qquad \vec{x} \quad = \vec{b}$$

$$A^TA = \begin{pmatrix} n & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{pmatrix} \qquad A^T\vec{b} = \begin{pmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{pmatrix}$$

Solve

$$(A^TA)\begin{pmatrix} C \\ D \\ E \end{pmatrix} = A^T\vec{b}$$

by row-reduction to get $\begin{pmatrix} C \\ D \\ E \end{pmatrix}$

Population of Austin over time

Date	Date-1839	Age^2	Age^3	Age^4	ln(age)	ln(age)^2
1850	11	121	1331	14641	2.397895273	5.749901739
1860	21	441	9261	194481	3.044522438	9.269116874
1870	31	961	29791	923521	3.433987204	11.79226812
1880	41	1681	68921	2825761	3.713572067	13.79061749
1890	51	2601	132651	6765201	3.931825633	15.45925281
1900	61	3721	226981	13845841	4.110873864	16.89928393
1910	71	5041	357911	25411681	4.262679877	18.17043973
1920	81	6561	531441	43046721	4.394449155	19.31118337
1930	91	8281	753571	68574961	4.510859507	20.34785349
1940	101	10201	1030301	104060401	4.615120517	21.29933738
1950	111	12321	1367631	151807041	4.709530201	22.17967472
1960	121	14641	1771561	214358881	4.795790546	22.99960696
1970	131	17161	2248091	294499921	4.875197323	23.76754894
1980	141	19881	2803221	395254161	4.94875989	24.49022445
1990	151	22801	3442951	519885601	5.017279837	25.17309696
2000	161	25921	4173281	671898241	5.081404365	25.82067032
Sum	1376	152336	18948896	2513367056	67.8437477	296.5200773
2010	171	29241			5.141663557	

pop	Age*pop	Age^2*pop	ln(pop)	Age*ln(pop)	ln(age)*ln(pop	Date
629	6919	76109	6.444131257	70.88544382	15.45235188	1850
3494	73374	1540854	8.158802491	171.3348523	24.83965725	1860
4428	137268	4255308	8.395703294	260.2668021	28.83073768	1870
11013	451533	18512853	9.306831672	381.5800986	34.56159013	1880
14575	743325	37909575	9.587063011	488.9402136	37.69466009	1890
22258	1357738	82822018	10.01045677	610.6378632	41.15172512	1900
29860	2120060	150524260	10.30427507	731.60353	43.92382599	1910
34876	2824956	228821436	10.45955419	847.2238896	45.96397908	1920
53120	4833920	439886720	10.88030878	990.1080994	49.07954431	1930
87930	8880930	896973930	11.38429632	1149.813929	52.53989953	1940
132459	14702949	1632027339	11.79402844	1309.137157	55.54433315	1950
186545	22571945	2731205345	12.13642778	1468.507761	58.20376558	1960
251808	32986848	4321277088	12.43642217	1629.171304	60.63001208	1970
345890	48770490	6876639090	12.75387608	1798.296528	63.11587041	1980
465622	70308922	10616647222	13.05112943	1970.720543	65.48116851	1990
656562	105706482	17018743602	13.39477241	2156.558358	68.06425498	2000
2301069	316477659	45057862749	170.4980792	16034.78637	745.0773758	Sum
790390						2010

Best line	Best parabola	Best power	Best exponent	Actual	Date
-117767	50896.99	422.3888104	2059.038293	629	1850
-82887	18307.79	2128.43188	3082.494049	3494	1860
-48007	-4643.41	5637.478458	4614.663844	4428	1870
-13127	-17956.61	11343.88264	6908.406653	11013	1880
21753	-21631.81	19580.43381	10342.26633	14575	1890
56633	-15669.01	30640.73766	15482.94392	22258	1900
91513	-68.21	44790.55349	23178.8222	29860	1910
126393	25170.59	62274.51176	34699.97705	34876	1920
161273	60047.39	83320.49114	51947.7822	53120	1930
196153	104562.19	108142.6618	77768.69914	87930	1940
231033	158714.99	136943.7036	116424.0379	132459	1950
265913	222505.79	169916.483	174293.2152	186545	1960
300793	295934.59	207245.3569	260926.5697	251808	1970
335673	379001.39	249107.2112	390621.4864	345890	1980
370553	471706.19	295672.3031	584781.94	465622	1990
405433	574048.99	347104.9563	875450.8629	656562	2000
4643353	832772.84			2301069	Sum
440313	686029.79	403564.1423	1310598.295	790390	2010

Best linear fit for Austin

$X =$ age $=$ date $- 1839$

$Y =$ population.

$$A^TA = \begin{pmatrix} n & \Sigma x \\ \Sigma x & \Sigma x^2 \end{pmatrix} = \begin{pmatrix} 16 & 1376 \\ 1376 & 152{,}336 \end{pmatrix}$$

$$A^Tb = \begin{pmatrix} \Sigma y \\ \Sigma xy \end{pmatrix} = \begin{pmatrix} 2{,}301{,}069 \\ 316{,}477{,}659 \end{pmatrix}$$

$$\begin{pmatrix} C \\ D \end{pmatrix} = (A^TA)^{-1}(A^Tb) = \begin{pmatrix} -156135 \\ 3488 \end{pmatrix}$$

Best fit is $\overset{y}{\text{population}} = 3488 \cdot (\text{age}) - 156135$

(lousy fit)

Parabolic fit for Austin

$$A^TA = \begin{pmatrix} n & \Sigma x & \Sigma x^2 \\ \Sigma x & \Sigma x^2 & \Sigma x^3 \\ \Sigma x^2 & \Sigma x^3 & \Sigma x^4 \end{pmatrix} = \begin{pmatrix} 16 & 1376 & 152336 \\ 1376 & 152336 & 18{,}948896 \\ 152336 & 18{,}948{,}896 & 2{,}513{,}367{,}056 \end{pmatrix}$$

$$A^Tb = \begin{pmatrix} \Sigma y \\ \Sigma xy \\ \Sigma x^2y \end{pmatrix} = \begin{pmatrix} 2301069 \\ 316477659 \\ 45{,}057{,}862{,}749 \end{pmatrix}$$

$$\begin{pmatrix} C \\ D \\ E \end{pmatrix} = \begin{pmatrix} 97877 \\ -4801 \\ 48.19 \end{pmatrix}$$

$$\text{Population} = 97{,}877 - 4{,}811 \cdot \text{Age} + 48.19\ \text{Age}^2$$

where $\text{Age} = \text{date} - 1839$

(still a poor fit)

Power law.

$$Y = C X^{\gamma}$$

$$\underline{\ln y} = \gamma \underline{\ln x} + \ln C$$

$$\begin{pmatrix} 1 & \ln X_1 \\ \vdots & \vdots \\ 1 & \ln X_n \end{pmatrix} \begin{pmatrix} \ln C \\ \gamma \end{pmatrix} = \begin{pmatrix} \ln Y_1 \\ \vdots \\ \ln Y_n \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 16 & \sum \ln X_i \\ \sum \ln X_i & \sum (\ln X_i)^2 \end{pmatrix} \qquad A^T b = \begin{pmatrix} \sum \ln Y_i \\ \sum \ln X_i \ln Y_i \end{pmatrix}$$

$\ln C = 0.04976 \Rightarrow C = 1.051$

$\gamma = 2.50135$

$$Y = (1.051)\, X^{2.50135}$$

(a little better)

$$y = c e^{kx}$$

$$\ln y = \ln c + kx$$

$$\begin{pmatrix} 1 & x_1 \\ 1 & \vdots \\ 1 & \vdots \\ 1 & x_n \end{pmatrix} \begin{pmatrix} \ln c \\ k \end{pmatrix} = \begin{pmatrix} \ln y_1 \\ \vdots \\ \vdots \\ \ln y_n \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 16 & \Sigma x \\ \Sigma x & \Sigma x^2 \end{pmatrix} \qquad A^T b = \begin{pmatrix} \Sigma \ln y \\ \Sigma x \ln y \end{pmatrix}$$

$$= \begin{pmatrix} 16 & 1376 \\ 1376 & 152{,}336 \end{pmatrix} \qquad = \begin{pmatrix} 170.49 \\ 16034.78\ldots \end{pmatrix}$$

$\ln c = 7.1858$ $\qquad c = e^{7.1858} \approx 1321$

$k = .04035$

$$\boxed{y = 1321\, e^{.04035x}}$$

Much better!