A $n \times n$ matrix A is Hermitian

if $\overline{A^T} = A$ $\qquad \overline{a+bi} = a-bi$

Ex: $\begin{pmatrix} 1 & 2+i \\ 2-i & 17 \end{pmatrix}$

Special case: If $A = \overline{A}$ and $A = A^T$, then A is Hermitian. (Real symmetric)

Thm: If H is Hermitian, then

1) E-vals of H are real. ✓
2) E-vecs w/ different e-vals are $\perp$. ✓

   If $H\vec{x} = \lambda_1 \vec{x}$ and $H\vec{y} = \lambda_2 \vec{y}$ with $\lambda_1 \neq \lambda_2$,

   then $\overline{x^T} y = 0$
3) H is diagonalizable. ✓ (handwave)

Cor: If A is real symmetric, then

1) E-vals of A are real,
2) E-vecs are real (nullspace of $A - \lambda I$)
3) E-vecs w/ different e-vals are $\perp$.
4) A is diagonalizable.
5) $A = R \Lambda R^{-1}$ with R orthogonal
   $(R^T = R^{-1})$

Ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $\qquad \lambda = 3, 1$

e-vecs $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ +1 \end{pmatrix}$

$$A = R \Lambda R^{-1} = R \Lambda R^{T}, \qquad R = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & +1/\sqrt{2} \end{pmatrix}$$

$$R = \begin{pmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin(\pi/4) \\ \sin\left(\frac{\pi}{4}\right) & \cos(\pi/4) \end{pmatrix} \quad \cancel{\Lambda =} \quad \Lambda = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

= rotates by 45°

$$H\vec{x} = \lambda\vec{x}$$

$$\overline{x^T} H\vec{x} = \overline{x^T}\lambda x = \lambda \, \overline{x^T} x \;\underline{=}\; \lambda$$

$$\overline{x^T} x = (\overline{x_1} \;\cdots\; \overline{x_n}) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{j=1}^{n} \overline{x_j}\, x_j = \sum_{j=1}^{n} |x_j|^2$$

$$> 0$$

scale $\vec{x}$ s.t. $\overline{x}^T x = 1$

$$\overline{x}^T H x = \lambda$$

$$x^T H^T \overline{x} = \lambda$$

$$\overline{x^T}\, \overline{H^T} x = \overline{\lambda}$$

$$\overline{x^T} H x = \overline{\lambda}$$ but $\overline{x^T} H x = \lambda$, so $\lambda = \overline{\lambda}$

λ is <u>real</u>.

$$Hx = \lambda_1 x$$

$$Hy = \lambda_2 y$$

$$\overline{X^T} H y = \overline{X^T} \lambda_2 y = \lambda_2 \overline{x^T} y$$

$$\begin{aligned}\overline{X^T} H y = \left(\overline{\overline{H^T} x}\right)^T y &= \left(\overline{Hx}\right)^T y \\ &= \overline{(\lambda_1 x)}^T y \\ &= \overline{\lambda_1}\, \overline{x^T} y \\ &= \lambda_1 \overline{x^T} y\end{aligned}$$

$$\lambda_2 \overline{x^T} y = \lambda_1 \overline{x^T} y$$

$$(\lambda_2 - \lambda_1) \overline{x^T} y = 0 \implies \overline{x^T} y = 0$$

$$a x_1^2 + 2b x_1 x_2 + c x_2^2$$

$$= (x_1 \; x_2) \begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{aligned} a x_1^2 &+ 2b\, x_1 x_2 + 2c\, x_1 x_3 \\ &+ d x_2^2 + 2e\, x_2 x_3 \\ &+ f x_3^2 \end{aligned} = (x_1 \; x_2 \; x_3) \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

For every symmetric matrix A there is a quadratic function $f_A(\vec{x}) = x^T A x$

$$\frac{\partial^2 f_A}{\partial x_i x_j} = A_{ij} + A_{ji} = 2A_{ij}$$

E-vals of A are $\lambda_1, \dots \lambda_n$

e-vecs are $\vec{b}_1, \dots \vec{b}_n$, s.t. ~~$\vec{b}_i \cdot \vec{b}_j =$~~

$$\vec{b}_i \cdot \vec{b}_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\vec{X} = \sum c_i \vec{b}_i \qquad A\vec{x} = \sum c_i A\vec{b}_i = \sum c_i \lambda_i \vec{b}_i$$

$$f_A(x) = X^T A x = \vec{X} \cdot \left(\sum c_i \lambda_i \vec{b}_i\right)$$

$$= \left(\sum_j c_j \vec{b}_j\right) \cdot \left(\sum_i c_i \lambda_i \vec{b}_i\right)$$

$$= \sum_{ij} c_i c_j \lambda_i \vec{b}_j \cdot \vec{b}_i = \sum_i c_i^2 \lambda_i$$

We say a symmetric matrix is positive-definite if all e-vals are >0.

Ex: $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ $\lambda = 3, 1,$ positive matrix

$$f_A(x) = 2X_1^2 - 2X_1X_2 + 2X_2^2 \geq 0$$

$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $\quad \lambda_1+\lambda_2=2$

$\lambda_1 \lambda_2 = -3 \qquad$ can't have $\lambda_1, \lambda_2 > 0$

not a positive matrix.

$f_A = x^2 + 4xy + y^2$ can be $<$.

$f_A \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1^2 - 4 + 1^2 = -2 < 0$

<u>Thm</u> A 2×2 matrix is positive if and only if it meets 2 tests.

1) $\det A > 0$
2) $\mathrm{Tr}\, A > 0$

<u>pf</u> If A is positive, $\lambda_1 > 0$, $\lambda_2 > 0$, so $\lambda_1 \lambda_2 > 0$

$\lambda_1 + \lambda_2 > 0$ ✓

If $\det A > 0$, $\lambda_1 \lambda_2 > 0$, λ_1, λ_2 have same sign. Trace tells whether both > 0 or both < 0.

$$ax^2+bx+c = \begin{pmatrix} x & 1 \end{pmatrix} \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}$$

$$\det = ac - \frac{b^2}{4} = -\frac{1}{4}\left(b^2-4ac\right)$$

<u>Thm</u> $\cancel{\begin{pmatrix} a & b \\ b & c \end{pmatrix}}$ A 2×2 real symmetric matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is positive if

1) $\det > 0$ and

2) $a > 0$

Similar matrices.

<u>Def</u> A and B are <u>similar</u> if

$$A = M^{-1} B M$$

$$(B = M A M^{-1})$$

$$(MA = BM).$$

Ex: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Thm If $A = M^{-1}BM$, then

$$\{\text{e-vals of } A\} = \{\text{e-vals of } B\}$$

$$\{\text{e-vecs of } A\} = M^{-1}\{\text{e-vecs of } B\}.$$

$$\left(\{\text{e-vecs of } B\} = M^{+1}\{\text{e-vecs of } A\}\right)$$

Pf

Suppose

$\cancel{A x}$ $$B\vec{x} = \lambda \vec{x}$$

$$A(M^{-1}\vec{x}) = M^{-1}BM(M^{-1}x) = M^{-1}Bx$$

$$= M^{-1}\lambda x = \lambda(M^{-1}x)$$

Other proof:

$$B = S\Lambda S^{-1}$$

$$A = M^{-1}S\Lambda S^{-1}M = (M^{-1}S)\Lambda(M^{-1}S)^{-1}$$

A and B have same 1) e-vals, 2) rank, 3) Trace, 4) Det.

but different 1) e-vecs 2) 4 Fundamental subspaces