Defn.

A is an $n \times n$ matrix.

$\vec{x} \in \mathbb{R}^n$

If $A\vec{x} = \lambda \vec{x}$ and $\vec{x} \neq \vec{0}$.

We say λ is an eigenvalue (e-val)

$\vec{x}$ is an eigenvector (e-vec)

How to find e-vals, ~~e-vecs~~

$\{\text{E-vals of } A\} = \{\text{roots of } \det(A - \lambda I)\}$.

Characteristic polynomial of A, $p_A(\lambda) = \det(\lambda I - A)$

$$= (-1)^n \det(A - \lambda I)$$

ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $A - \lambda I = \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}$

$$p_A(\lambda) = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3$$

Roots $\lambda = 3, 1$ = e-vals.

How to find e-vecs.

E-vecs with e-val λ are solns to $(A-\lambda I)\vec{x}=0$.

Eigenspace with e-val λ = $\{$ Soln to $(A-\lambda I)x=0 \}$.
$(E_\lambda =)$

$$= \text{Nul}(A-\lambda I).$$

Get soln by row-reducing.

E_3: $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ $\quad$ $\begin{aligned} x_1 &= x_2 \\ x_2 &= x_2 \end{aligned}$ $\quad$ $\vec{x} = c\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

E_1: $\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad$ $\begin{aligned} x_1 &= -x_2 \\ x_2 &= x_2 \end{aligned}$ $\quad$ $\vec{x} = c\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad A - \lambda I = \begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix}$$

$$\det(A-\lambda I) = (1-\lambda)(2-\lambda)(1-\lambda) - (2-\lambda)$$
$$= (2-\lambda)\left[(1-\lambda)^2 - 1\right]$$
$$= (2-\lambda)(\lambda^2 - 2\lambda) = -\lambda(\lambda-2)^2$$

E_2: $\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$x_1 = x_3$$
$$x_2 = x_2$$
$$x_3 = x_3$$

E_2 is 2-dim'l.

$$\text{basis} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$E_0 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$x_1 = -x_3$$
$$x_2 = 0$$
$$x_3 = x_3$$

E_0 is 1 dim'l, basis $= \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

Goal: Find basis $\{\vec{x}_1, \dots \vec{x}_n\}$ of e-vecs.

e-vals $\{\lambda_1, \dots \lambda_n\}$

$$S = \begin{pmatrix} \vec{x}_1, \dots & \vec{x}_n \end{pmatrix}$$

matrix of e-vecs.
(aka P)

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

(aka D)

Claim: $A = S \Lambda S^{-1} = PDP^{-1}$

Equivalently $AS = S\Lambda$

$$AS = A \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} = \begin{pmatrix} A\vec{x}_1, & \cdots & A\vec{x}_n \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 \vec{x}_1 & \cdots & \lambda_n \vec{x}_n \end{pmatrix} = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} = S\Lambda$$

$$A^m = (S\Lambda S^{-1})^m = S\Lambda \underbrace{S^{-1} S}\Lambda \underbrace{S^{-1} \cdots S}\Lambda S^{-1}$$

$$= S \Lambda I \Lambda I \Lambda \cdots \Lambda S^{-1}$$

$$= S \Lambda^m S^{-1}$$

$$= S \begin{pmatrix} \lambda_1^m & & 0 \\ & \ddots & \\ 0 & & \lambda_n^m \end{pmatrix} S^{-1}$$

$$\vec{X}(m+1) = A\,\vec{x}(m)$$

$$\vec{x}(m) = A^m \vec{x}(0)$$

$$= S A^m S^{-1} \vec{x}(0)$$

$$\begin{tikzcd}
\vec{y}(0) \arrow[r, "\Lambda^m", "\text{one variable at a time}"'] & \vec{y}(m) \arrow[d, "S"] \\
\vec{X}(0) \arrow[u, "S^{-1}"] \arrow[r, "A^m"] & \vec{x}(m)
\end{tikzcd}$$

Basis $\{b_1, \ldots b_n\}$ of e-vecs.

$$\vec{x} = y_1 \vec{b}_1 + y_2 \vec{b}_2 + \cdots + y_n \vec{b}_n$$

$$S = \left(\vec{b}_1 \cdots \vec{b}_n \right) \qquad \vec{x} = S \vec{y}$$

$$\vec{y} = S^{-1} \vec{x}$$

$$\vec{x}(m+1) = A \vec{x}(m)$$

$$S \vec{y}(m+1) = S \Lambda S^{-1} S \vec{y}(m)$$

$$= S \Lambda \vec{y}(m)$$

$$\vec{y}(m+1) = \Lambda \vec{y}(m)$$

decoupled equations!

$$y_k(m) = \lambda_k^m y_k(0)$$

$$\vec{y}(m) = \Lambda^m \vec{y}(0)$$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$y_1 = x_1 + x_2$$
$$y_2 = x_1 - x_2$$

$$\vec{y} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \vec{x}$$

$$x_1 = \frac{y_1 + y_2}{2}$$

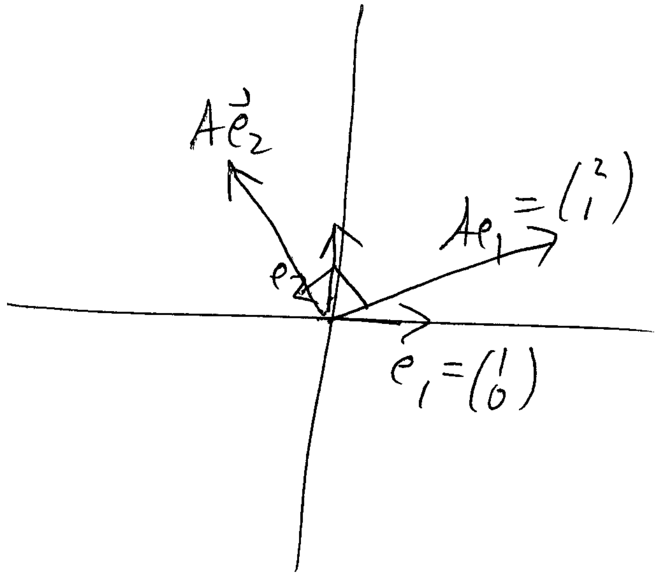
$$S^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$x_2 = \frac{y_1 - y_2}{2} \qquad S = \frac{1}{-2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

$$\vec{x} = S \vec{y}$$

$$\vec{b}_1 = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \qquad \vec{b}_2 = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$



$$A - \lambda I = \begin{pmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{pmatrix}$$

$$\det = (2-\lambda)^2 + 1$$

$$= \lambda^2 - 4\lambda + 5$$

$$\lambda = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm \sqrt{-4}}{2}$$

$$= 2 \pm i$$

E_{2+i} $\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}$

$x_1 = i x_2$

$x_2 = x_2$

e-vec: $\begin{pmatrix} i \\ 1 \end{pmatrix}$

E_{2-i} $\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}$

$x_1 = -i x_2$

$x_2 = x_2$

$\begin{pmatrix} -i \\ 1 \end{pmatrix}$

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} \qquad (A - \lambda I) = \begin{pmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (3-\lambda)^2 = (\lambda-3)^2$$

$$\lambda = 3.$$

$$E_3: \; A - 3I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \begin{aligned} x_1 &= x_1 \\ x_2 &= 0 \end{aligned}$$

$\lambda = 3$ has algebraic multiplicity 2 since 3 is double root of $(\lambda-3)^2$

$\lambda = 3$ has geometric multiplicity 1 since $\dim E_3 = 1$.

$\sum$ algebraic multiplicities $= n$. $\qquad 1 \leq GM \leq AM$

$\sum$ geometric multiplicities $\leq n$.

If $AM \neq GM$ for some λ, A is not diagonalizable

Tricks of the trade

1) $\mathrm{Trace}(A) = \sum$ e-vals.

Def $\mathrm{Tr}(A) = \sum$ diagonal entries.

$$\mathrm{Tr}(AB) = \sum_i (AB)_{ii} = \sum_{ij} (A_{ij} B_{ji})$$

$$= \sum_{ij} B_{ji} A_{ij} = \sum_j (BA)_{jj} = \mathrm{Tr}(BA)$$

$$\mathrm{Tr}\,(S\Lambda S^{-1}) = \mathrm{Tr}\,(\Lambda S^{-1} S) = \mathrm{Tr}\,\Lambda = \lambda_1 + \dots + \lambda_n$$

e.g. $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $\quad$ $\mathrm{Tr}(A) = 4$

e-vals $= 3, 1,$ $\quad$ $3 + 1 = 4.$

2) $\mathrm{Det}(A) = \prod$ e-vals.

$$\det(A) = \det(S\Lambda S^{-1}) = \underline{\det(S)}\, \det(\Lambda)\, \underline{\det(S^{-1})}$$
$$= \det \Lambda = \lambda_1 \cdots \lambda_n$$

$$A = \begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix}$$

$\mathrm{Tr}(A) = 0$

$\det(A) = -25$

$\lambda = 5, -5$

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3) Scaling & adding identity

$$B \cancel{A} = \begin{pmatrix} 40 & 30 \\ 30 & -40 \end{pmatrix} \qquad \lambda = 50, -50$$

if $A\vec{x} = \lambda\vec{x}$ $\qquad (10A)\vec{x} = 10\lambda\,\vec{x}$

Rescaling matrix rescales e-vals, doesn't change e-vecs.

$$C = \begin{pmatrix} 1004 & 3 \\ 3 & 996 \end{pmatrix} = 1000I + A. \qquad \lambda = 1005, 995$$

$$Ax = \lambda x, \quad (1000I + A)\vec{x} = 1000\vec{x} + \lambda\vec{x} = (1000+\lambda)\vec{x}$$

$$D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \lambda = 1, -1$$

$$\text{e-vecs} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ +1 \end{pmatrix}$$

$$\begin{pmatrix} 13 & 35 \\ 35 & 13 \end{pmatrix} \qquad \lambda = 13 \pm 35$$

$$\text{e-vecs} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= 13I + 35D$$

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} \qquad \lambda = a \pm b$$

$$\text{e-vecs} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= bD + aI$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \lambda = i, -i$$

$$\text{e-vecs} \begin{pmatrix} i \\ 1 \end{pmatrix}, \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \qquad \lambda = a \pm bi$$

$$\text{e-vecs} \begin{pmatrix} i \\ 1 \end{pmatrix}, \begin{pmatrix} -i \\ 1 \end{pmatrix}$$