Tricks of the trade.

1) $\text{Tr}(A) \stackrel{\text{def}}{=} \sum A_{ii} = \sum_{i=1}^{n} \lambda_i$

(Counted w/ multiplicity)

2) $\det(A) = \prod_{i=1}^{n} \lambda_i$

3) If A has e-vals $\lambda_1, \dots \lambda_n$

e-vecs $\vec{x}_1, \dots \vec{x}_n$,

then $(aI + bA)$ has e-vals $\{a + b\lambda_i\}$

e-vecs $\vec{x}_i$

4) Row sums, column sums + transposes

$$\begin{pmatrix} 3 & 2 & 5 \\ 8 & 1 & 1 \\ 3 & 9 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix}$$

10 is e-val

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is e-vec.

If all rows add up to c, c is e-val

$\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ is e-vec

E-vals of A^T are same as e-vals of A.

(e-vecs usually different)

$$\det(A^T - \lambda I) = \det((A-\lambda I)^T) = \det(A-\lambda I)$$

$$\begin{pmatrix} 3 & 8 & 3 \\ 2 & 1 & 9 \\ 5 & 1 & -2 \end{pmatrix}$$

<u>columns</u> add up to 10

10 is e-val.

If columns of A add up to c

c is an e-val.

Digression on probability (8.3)

Streaky football team.

If won last game

$p(\text{win next}) = .9$

$p(\text{lose next}) = .1$

If lost last game,

$p(\text{win next}) = .2$

$p(\text{lose next}) = .8$

n	1	2	3
p(win)	.9	$(.9)(.9) + (.1)(.2) = .83$	$(.83)(.9) + (.17)(.2) = .781$
p(loss)	.1	$(.9)(.1) + (.1)(.8) = .17$	$(.83)(.1) + (.17)(.8) = .219$

$$\vec{X}(n) = \begin{pmatrix} p(\text{win } n\text{-th game}) \\ p(\text{lose } n\text{-th game}) \end{pmatrix}$$

$$\vec{X}(n+1) = \begin{pmatrix} .9 & .2 \\ .1 & .8 \end{pmatrix} \vec{X}(n)$$

A <u>Markov matrix</u> is a matrix w non-neg entries s.t. each column adds up to 1.

Biggest e-val is 1. other e-val $= .7$

E_1 : $\begin{pmatrix} -.1 & .2 \\ .1 & -.2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$

$$x_1 = 2x_2$$
$$x_2 = x_2$$

$$\vec{b}_1 = c\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$$

$=$ steady-state probability.

$E_{.7}$ $\quad \begin{pmatrix} .2 & .2 \\ .1 & .1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \vec{b}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\lambda_1 = 1 \quad \vec{b}_1 = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$

$\lambda_2 = .7 \quad \vec{b}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\vec{X}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \vec{b}_1 - \frac{1}{3} \vec{b}_2$

$\vec{y}(0) = \begin{pmatrix} 1 \\ -1/3 \end{pmatrix}$

$$\begin{tikzcd} \vec{y}(0) \arrow[r, "\Lambda^n"] & \vec{y}(m) \arrow[d, "S"] \\ \vec{X}(0) \arrow[u, "S^{-1}"] & \vec{X}(m) \end{tikzcd}$$

$\vec{y}(m) = \begin{pmatrix} 1 \cdot 1^m \\ -\frac{1}{3} \cdot (.7)^m \end{pmatrix}$

$\vec{X}(m) = 1 \cdot \vec{b}_1 - \frac{1}{3}(.7)^m \vec{b}_2$

$= \begin{pmatrix} \frac{2}{3} + \frac{1}{3}(.7)^m \\ 1/3 - \frac{1}{3}(.7)^m \end{pmatrix}$ converges to $\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$

as $m \to \infty$

Weather.

$$\underbrace{\begin{pmatrix} p(\text{sunny}) \\ p(\text{cloudy}) \\ p(\text{rainy}) \end{pmatrix}}_{P(m+1)} (m+1) = \begin{pmatrix} .8 & .2 & .3 \\ .1 & .5 & .2 \\ .1 & .3 & .5 \end{pmatrix} \begin{matrix} \cancel{p(\text{today})} \\ P(m) \end{matrix}$$

Markov matrix $\lambda = 1$.

$\cancel{E_1}$

$$E_1: \begin{pmatrix} -.2 & .2 & .3 \\ .1 & -.5 & .2 \\ .1 & .3 & -.5 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & -19/8 \\ 0 & 1 & -7/8 \\ 0 & 0 & 0 \end{pmatrix}$$

$$X_1 = \frac{19}{8} X_3$$

$$X_2 = \frac{7}{8} X_3$$

$$X_3 = X_3$$

$$\text{e-vec} = c \begin{pmatrix} 19/8 \\ 7/8 \\ 1 \end{pmatrix} = \begin{pmatrix} 19/34 \\ 7/34 \\ 8/34 \end{pmatrix}$$

Block triangular, block diagonal.

$$M = \left(\begin{array}{c|c} A & B \\ \hline O & C \end{array}\right)$$

A, C square

B rectangular.

$$\det(M-\lambda I) = \det \left(\begin{array}{c|c} A-\lambda I & B \\ \hline O & C-\lambda I \end{array}\right)$$

$$= \det(A-\lambda I)\ \det(C-\lambda I)$$

E-vals of M = {e-vals of A} $\cup$ {e-vals of C}.

(Same goes for $\left(\begin{array}{c|c} A' & O \\ \hline B' & C' \end{array}\right)$)

If $A\vec{x} = \lambda\vec{x}$

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}\begin{pmatrix} \vec{x} \\ \vec{0} \end{pmatrix} = \begin{pmatrix} A\vec{x} + B\vec{0} \\ 0\vec{x} + C\vec{0} \end{pmatrix} = \begin{pmatrix} \lambda\vec{x} \\ \vec{0} \end{pmatrix}$$

$$= \lambda\begin{pmatrix} \vec{x} \\ \vec{0} \end{pmatrix}$$

If $C\vec{y} = \lambda\vec{y}$ and λ is <u>not</u> an e-val of A, then you can find $\vec{x}$ s.t. $\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}$ is e-vec of M.

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} = \begin{pmatrix} A\vec{x} + B\vec{y} \\ C\vec{y} \end{pmatrix} = \begin{pmatrix} \stackrel{?}{=} \lambda\vec{x} \\ \lambda\vec{y} \end{pmatrix}$$

Need $A\vec{x} + B\vec{y} = \lambda\vec{x}$

$$B\vec{y} = (\lambda I - A)\vec{x} \quad ;$$

$$x = (\lambda I - A)^{-1} B\vec{y}$$

If $B=0$, can pad in either direction.

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & C \end{array}\right) \qquad \text{e-vecs } \begin{pmatrix} \vec{x} \\ \vec{0} \end{pmatrix}, \begin{pmatrix} \vec{0} \\ \vec{y} \end{pmatrix}$$

block diagonal

$$M = \begin{array}{c} A \\ \left(\begin{array}{cc|ccc} 5 & 3 & 3 & 0 & 4 \\ -3 & -5 & -2 & 7 & 1 \\ \hline 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right) \\ C \end{array}$$

e-vals of M = (evals of A) $\cup$ (e-vals of C)

$\mathrm{Tr}(A)=0$, $\det(A)=-16$, e-vals of A are ± 4.

Cols of C add to 3

$\mathrm{Tr}=6$

$\det=6$

$\lambda=1,2,3$

Problems you can solve w/ diagonalization

1) $\vec{x}(n+1) = A\vec{x}(n)$ (e.g. probability, A = Markov)

2) Coupled systems of 1st order ODE's

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

3) Couples ~~ys~~ systems of 2nd order ODE's

$$\frac{d^2\vec{x}}{dt^2} = A\vec{x}$$

Method of attack.

1) Diagonalize A. $A = S \Lambda S^{-1}$

$$S = \begin{pmatrix} \vec{b}_1 & \cdots & \vec{b}_n \end{pmatrix} = \text{e-vecs}$$

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = \text{e-vals.}$$

2) Write $\vec{x} = y_1 \vec{b}_1 + \cdots + y_n \vec{b}_n$

$$\vec{x} = S\vec{y} \quad ; \quad \vec{y} = S^{-1}\vec{x}$$

3) Rewrite equations in terms of y.

They $\boxed{\text{decouple}}$

$$y_i(n+1) = \lambda_i \, y_i(n)$$

or $$\frac{dy_i}{dt} = \lambda_i \, y_i$$

or $$\frac{d^2 y_i}{dt^2} = \lambda_i \, y_i$$

4) Solve one variable at a time.

5) Convert back to $\vec{x}$

$$\begin{tikzcd}
\vec{y}(0) \arrow[rr, "\text{one at a time}"] & & \vec{y}(\text{later}) \arrow[d, "S"] \\
\vec{X}(0) \arrow[u, "S^{-1}"] & & \vec{X}(\text{later})
\end{tikzcd}$$