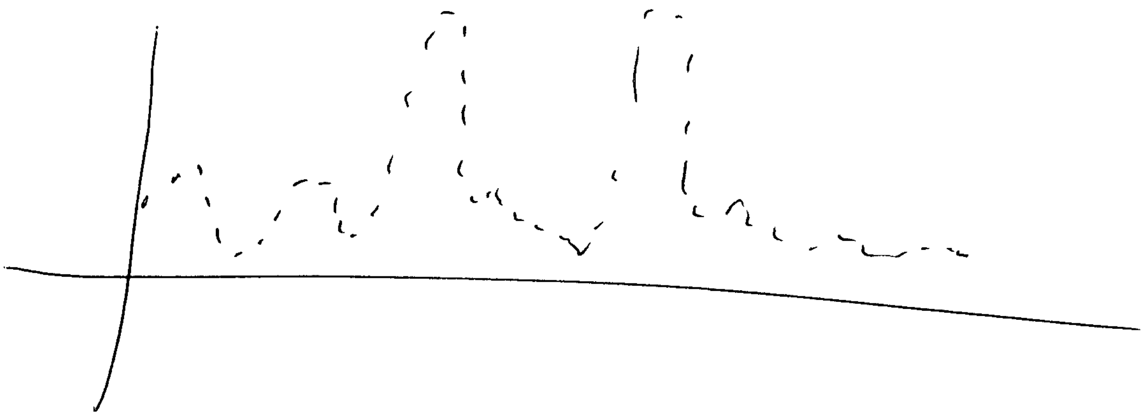
Signal Processing

later time,

Step 1: Sample at N points. $\Delta t = \frac{\text{total time}}{N}$



Get vector $\vec{V}(n) =$ signal at time $n\Delta t$

$\in \mathbb{R}^N$

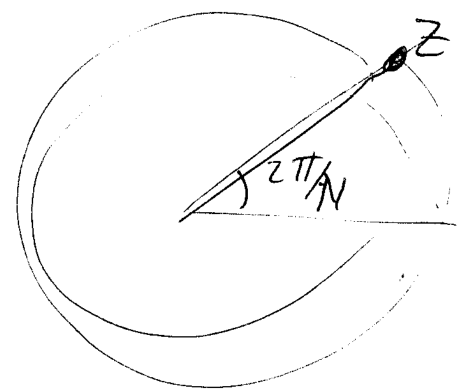
Obvious basis for $\mathbb{R}^n$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ \vdots \end{pmatrix} \qquad e_k(j) = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise.} \end{cases}$$

$\mathcal{E}$ = standard basis.

$\mathcal{E}$ is a good basis for spiky localized data. (E.g. stock prices).

Fourier basis.



unit circle in $\mathbb{C}$

$$z = e^{2\pi i/N}$$

$$= \cos\left(\frac{2\pi}{N}\right) + i\sin\left(\frac{2\pi}{N}\right)$$

$$f_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$f_2 = \begin{pmatrix} 1 \\ z \\ z^2 \\ \vdots \\ z^{N-1} \end{pmatrix}$$

$$f_k(n) = z^{(k-1)(n-1)}$$

$$f_3 = \begin{pmatrix} 1 \\ z^2 \\ z^4 \\ \vdots \\ z^{2(N-1)} \end{pmatrix}$$

$$\vec{v} = \begin{pmatrix} 1 \\ 3 \\ -2 \\ 5 \end{pmatrix} \qquad [v]_F = P_{FE} [\vec{v}]_E$$

$$= \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -2 \\ 5 \end{pmatrix}$$

Change-of-basis in obvious way takes N^2 steps.

Fast Fourier Transform (FFT) does it in $N \ln N$

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discrete derivative $(Dv)(n) = v(n+1) - v(n) \quad (n<N)$

$$(Dv)(N) = v(1) - v(N)$$

$$D \begin{pmatrix} 1 \\ z^k \\ z^{2k} \\ \vdots \\ z^{(N-1)k} \end{pmatrix} = (z^k - 1) \begin{pmatrix} 1 \\ z^k \\ \vdots \\ z^{(N-1)k} \end{pmatrix}.$$

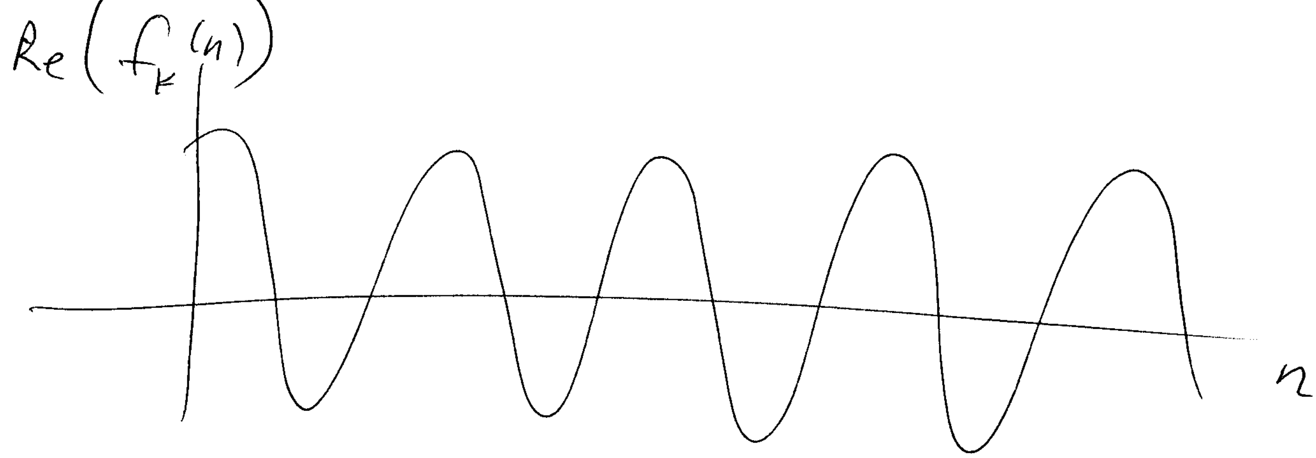
E-vecs = Fourier basis.

Fourier basis (aka "frequency domain")

is great for anything involving

derivatives (e.g. physics, aka real world).

Problem: Not localized in time.



Uncertainty principle $\doteq$

localized in time $\longrightarrow$ not localized in frequency

localized in frequency $\longrightarrow$ not localized in time.

$$\boxed{\Delta t \, \Delta \text{freq} \gtrsim 1.}$$

$$N=4$$

$$z = e^{2\pi i/4} = e^{\pi i/2} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$$

$$f_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$f_2 = \begin{pmatrix} 1 \\ i \\ i^2 \\ i^3 \end{pmatrix} = \begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix}$$

$$f_3 = \begin{pmatrix} 1 \\ i^2 \\ i^4 \\ i^6 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$f_4 = \begin{pmatrix} 1 \\ i^3 \\ i^6 \\ i^9 \end{pmatrix} = \begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix}$$

$$P_{EF} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

$$P_{FE} = P_{EF}^{-1} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & +i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} = \frac{1}{N} \overline{P_{EF}}^{T}$$

Wavelets

$N=8$.

$$w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad w_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \quad w_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad w_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

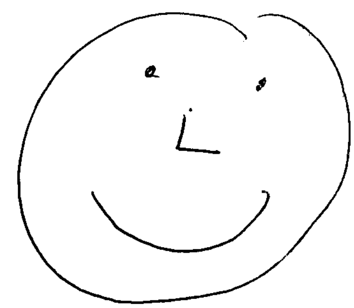
$$w_5 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad w_6 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad w_7 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad w_8 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

Basic graphics in $\mathbb{R}^2$

Points $= \begin{pmatrix} x \\ y \end{pmatrix}$

Rotation by θ: $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

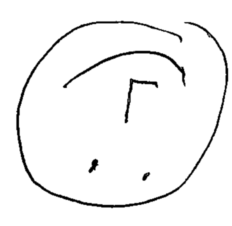
Enlarge by 2: $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$



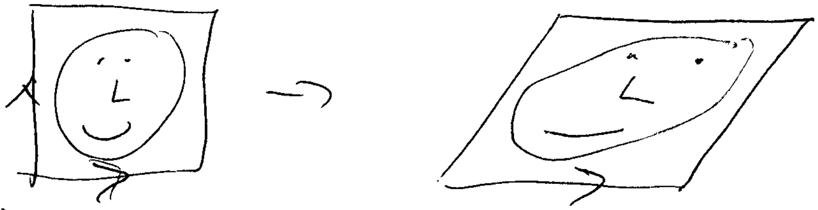
Stretch in x-direction $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

Reflect in x direction $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Reflect vertically $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



Shear



$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Stretch by 3 in direction 37° ccw of x-axis

$$R_{37^\circ} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} R_{-37^\circ}$$

To incorporate motion.
add a 3rd variable

$$(x,y) \longrightarrow \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Rotate :
$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Other linear maps, use
$$\left(\begin{array}{c|c} M & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline 0 \; 0 & 1 \end{array}\right)$$
instead of M.

$$\left(\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \\ \hline 0 & 0 & 1 \end{array}\right) \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x+a \\ y+b \\ 1 \end{pmatrix}$$

adds (a,b) to (x,y)

What does

$$A = \left(\begin{array}{cc|c} 2 & 0 & 5 \\ 0 & -1 & 3 \\ \hline 0 & 0 & 1 \end{array}\right)$$

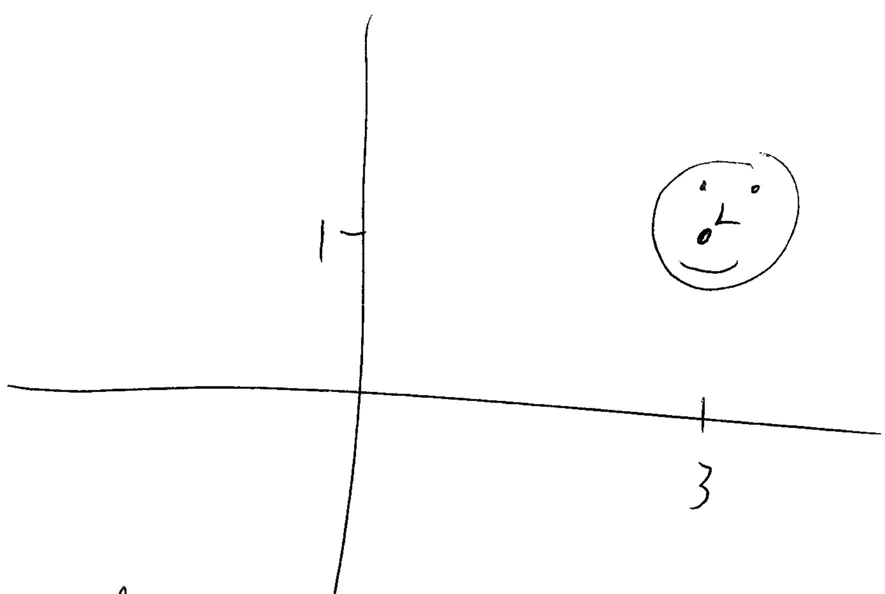
do?

$$A\begin{pmatrix}x\\y\\1\end{pmatrix} = \begin{pmatrix}2x+5\\-y+3\\1\end{pmatrix}$$

doubles x

flips sign of y

adds $\begin{pmatrix}5\\3\end{pmatrix}$



How do I rotate it by θ while keeping it at $(3,1)$?

Ans:

$$\begin{pmatrix}1 & 0 & 3\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\left(\begin{array}{cc|c}1 & 0 & -3\\ 0 & 1 & -1\\ 0 & 0 & 1\end{array}\right)$$

3D graphics

locations: $\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$

Everything is a 4x4 matrix.

Move by (a, b, c) : $T_{(a,b,c)} = \left(\begin{array}{c|c} I & \begin{matrix} a \\ b \\ c \end{matrix} \\ \hline 0\ 0\ 0 & 1 \end{array}\right)$

Rotate ~~in~~ about x-axis

$$R_{1,\theta} = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

$$R_{2,\theta} = \left(\begin{array}{ccc|c} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

$$R_{3,\theta} = \left(\begin{array}{ccc|c} \cos\theta & -\sin\theta & 0 & 0 \\ +\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$