M340L Final Exam Solutions, May 10, 2012

1) Consider the matrix $A = \begin{pmatrix} 1 & 3 & -2 & 5 \\ 2 & 6 & -2 & 14 \\ -1 & -3 & 8 & 12 \end{pmatrix}$.

a) Put A in row-echelon form using only the operations of the forward phase of row-reduction. (That is, don't flip rows unless necessary, don't rescale rows, and don't subtract multiples of a later row from an earlier row.)

Subtract twice R1 from R2, add R1 to R3, and then subtract 3 R2 from R3 to get $U = \begin{pmatrix} 1 & 3 & -2 & 5 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 5 \end{pmatrix}$.

b) Write A as a product LU, where L is lower-triangular with 1's on the diagonal and U is in row-echelon form.

U is as before, while $L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{pmatrix}$ records the steps of the row reduction.

c) Find the *reduced* row-echelon form of A.

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2) Consider the matrix $A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$.

First row reduce A and A^T to get $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$, respectively.

a) Are the columns of A linearly independent? Do they span $\mathbb{R}^3$?

Yes and no. There's a pivot in each column, but not in each row.

b) Are the columns of A^T linearly independent? Do they span $\mathbb{R}^2$?

No and yes. There's a pivot in each row but not in each column.

c) Find bases for the four fundamental subspaces of A.

For $Col(A)$, the pivot columns of A: $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}$

For $Nul(A)$, the empty set. (There are no free variables)

For $Row(A)$, $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

For $Nul(A^T)$, the single vector $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, obtained by solving $A^T\mathbf{x} = 0$.

3) The matrix $[A|\mathbf{b}] = \begin{pmatrix} 1 & 1 & 1 & 1 & 6 & | & 6 \\ 1 & 1 & 1 & -1 & 2 & | & 4 \\ 1 & -1 & 3 & 0 & -6 & | & -3 \\ 2 & 0 & 4 & -1 & -4 & | & 1 \end{pmatrix}$ is row-equivalent to $\begin{pmatrix} 1 & 0 & 2 & 0 & -1 & | & 1 \\ 0 & 1 & -1 & 0 & 5 & | & 4 \\ 0 & 0 & 0 & 1 & 2 & | & 1 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$.

a) Find all solutions to $A\mathbf{x} = \mathbf{b}$.

Writing our the row-reduced equations and throwing in the "dummy" equations $x_3 = x_3$ and $x_5 = x_5$ yields: $\mathbf{x} = \begin{pmatrix} 1 \\ 4 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -5 \\ 0 \\ -2 \\ 1 \end{pmatrix}$.

b) Find all vectors in $\mathbb{R}^5$ that are orthogonal to $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 3 \\ 0 \\ -6 \end{pmatrix}$, and $\begin{pmatrix} 2 \\ 0 \\ 4 \\ -1 \\ -4 \end{pmatrix}$.

The orthogonal complement of the row space is the null space of A, which is the span of $\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -5 \\ 0 \\ -2 \\ 1 \end{pmatrix}$.

4a) Find a least-squares solution to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 7 \end{pmatrix}$.

We compute $A^T A = \begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix}$ and $A^T\mathbf{b} = \begin{pmatrix} 14 \\ 45 \end{pmatrix}$, so $\mathbf{x} = (A^T A)^{-1} A^T \mathbf{b} = \begin{pmatrix} -3/2 \\ 2 \end{pmatrix}$.

b) Find the point in the column space of A that comes closest to $\mathbf{b}$.

This is $A\mathbf{x}$, with $\mathbf{x} = \begin{pmatrix} -3/2 \\ 2 \end{pmatrix}$, namely $\begin{pmatrix} 1/2 \\ 5/2 \\ 9/2 \\ 13/2 \end{pmatrix}$.

c) Find the equation of the best line "through" the points (1,1), (2,2), (3,4) and (4,7).

This is an application solved by the least-squares calculation of part (a), with the result $y = 2x - 3/2$.

5a) Use Gram-Schmidt to find an orthogonal basis for the column space of $\begin{pmatrix} 1 & 2 & 6 \\ 1 & 2 & 4 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

$$\mathbf{y}_1 = \mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathbf{y}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{y}_1}{\mathbf{y}_1 \cdot \mathbf{y}_1}\mathbf{y}_1 = \mathbf{x}_2 - \frac{4}{4}\mathbf{y}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$\mathbf{y}_3 = \mathbf{x}_3 - \tfrac{\mathbf{x}_3\cdot\mathbf{y}_1}{\mathbf{y}_1\cdot\mathbf{y}_1}\mathbf{y}_1 - \tfrac{\mathbf{x}_3\cdot\mathbf{y}_2}{\mathbf{y}_2\cdot\mathbf{y}_2}\mathbf{y}_2 = \mathbf{x}_3 - 3\mathbf{y}_1 - 2\mathbf{y}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

b) Let $\mathbf{v} = \begin{pmatrix} 3 \\ 5 \\ -2 \\ -2 \end{pmatrix}$. This vector is in the column space of A. Find its coordinates with respect to the basis you found in part (a).

Since the basis is orthogonal, we can find coefficients by taking inner products. If $\mathbf{v} = \sum c_i\mathbf{y}_i$, then $c_i = \mathbf{x}\cdot\mathbf{y}_i/(\mathbf{y}_i\cdot\mathbf{y}_i)$. This gives $c_1 = 4/4 = 1$, $c_2 = 12/4 = 3$ and $c_3 = -2/2 = -1$, so $[\mathbf{v}]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$.

6a) Let $L_1 : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation given by the formula $L_1(\mathbf{x}) = \begin{pmatrix} 3x_1 - x_3 \\ x_2 + x_3 \end{pmatrix}$. Find the matrix of L_1 (with respect to the standard bases for $\mathbb{R}^3$ and $\mathbb{R}^2$).

$$[L_1] = (\ L(e_1)\ L(e_2)\ L(e_3)\ ) = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

b) Let V be the space of quadratic polynomials in a variable t, and let $\mathcal{B} = \{1, t, t^2\}$ be a basis for V. Let $L_2 : V \to V$ be a linear transformation given by the formula $L_2\mathbf{x}(t) = \mathbf{x}(2t) + \mathbf{x}(1)$. Find the matrix $[L_2]_{\mathcal{B}\mathcal{B}}$ of L_2 with respect to the basis $\mathcal{B}$.

Since $L_2(1) = 2$, $L_2(t) = 1 + 2t$ and $L_2(t^2) = 1 + 4t^2$, we have $[L]_{\mathcal{B}\mathcal{B}} = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

7a) In $\mathbb{R}^2$, consider the bases $\mathcal{E} = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ and $\mathcal{B} = \left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \end{pmatrix}\right\}$. Let $\mathbf{x} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Find the change-of-basis matrices $P_{\mathcal{E}\mathcal{B}}$ and $P_{\mathcal{B}\mathcal{E}}$, and compute $[\mathbf{x}]_{\mathcal{B}}$.

$P_{\mathcal{E}\mathcal{B}} = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$, whose inverse is $P_{\mathcal{B}\mathcal{E}} = \begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix}$, and $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}\mathcal{E}}\mathbf{x} =$

$\begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 13 \\ -5 \end{pmatrix}$.

b) Let V be the space of linear polynomials in a variable t, and consider the bases $\mathcal{E} = \{1, t\}$ and $\mathcal{B} = \{1+t, 4+3t\}$. Let $\mathbf{x}$ be the polynomial $8-t$. Find the change-of-basis matrices $P_{\mathcal{E}\mathcal{B}}$ and $P_{\mathcal{B}\mathcal{E}}$, and compute $[\mathbf{x}]_{\mathcal{B}}$.

This is similar. $P_{\mathcal{E}\mathcal{B}} = \begin{pmatrix} 1 & 4 \\ 1 & 3 \end{pmatrix}$, $P_{\mathcal{B}\mathcal{E}} = P_{\mathcal{E}\mathcal{B}}^{-1} = \begin{pmatrix} -3 & 4 \\ 1 & -1 \end{pmatrix}$, $[\mathbf{x}]_{\mathcal{E}} = \begin{pmatrix} 8 \\ -1 \end{pmatrix}$, and $[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} -3 & 4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 8 \\ -1 \end{pmatrix} = \begin{pmatrix} -28 \\ 9 \end{pmatrix}$.

8. Let $A = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$

a) Find the eigenvalues and eigenvectors of A.

Since the matrix is of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, the eigenvalues are $a \pm bi = -1 \pm i$ and the eigenvectors are $\begin{pmatrix} \pm i \\ 1 \end{pmatrix}$.

b) Find a solution to the system of differential equations $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ with initial condition $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$. Simplify your answer as much as possible. Describe what happens as $t \to \infty$. Is the system stable or unstable?

Since $\mathbf{x}(0) = \mathbf{b}_1 + \mathbf{b}_2$, $\mathbf{x}(t) = e^{(-1+i)t}\mathbf{b}_1 + e^{(-1-i)t}\mathbf{b}_2$. Since $e^{(-1\pm i)t} = e^{-t}(\cos(t) + i\sin(t))$, this works out to $\mathbf{x}(t) = 2e^{-t}\begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}$. As $t \to \infty$, $\mathbf{x}$ both oscillates and shrinks, approaching zero in the limit. The system is stable.

c) Find a solution to the system of discrete-time evolution equations $\mathbf{x}(n+1) = A\mathbf{x}(n)$ with initial condition $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$. Describe what happens as $n \to \infty$. Is the system stable or unstable?

Since $\mathbf{x}(0) = \mathbf{b}_1 + \mathbf{b}_2$, $\mathbf{x}(n) = (-1+i)^n\mathbf{b}_1 + (-1-i)^n\mathbf{b}_2$. This doesn't simplify as neatly as in part (b) (and you were not asked to simplify). For those who are interested, the powers of $-1 \pm i$ go as follows: $(-1 \pm i)^2 = \mp 2i$, $(-1 \pm i)^3 = 2(1 \pm i)$, $(-1 \pm i)^4 = -4$, and $(-1 \pm i)^{n+4} = -4(-1 \pm i)^n$. This gives

$$\mathbf{x}(n) = \begin{cases} (-4)^k \begin{pmatrix} 0 \\ 2 \end{pmatrix} & \text{if n=4k} \\ (-4)^k \begin{pmatrix} -2 \\ -2 \end{pmatrix} & \text{if n=4k+1} \\ (-4)^k \begin{pmatrix} 4 \\ 0 \end{pmatrix} & \text{if n=4k+2} \\ (-4)^k \begin{pmatrix} -4 \\ 4 \end{pmatrix} & \text{if n=4k+3} \end{cases}$$

As $n \to \infty$, $\mathbf{x}$ both oscillates and grows, since both $-1+i$ and $-1-i$ have magnitude bigger than 1. The system is unstable.

9a. In $\mathbb{R}^3$, find a matrix R_x that rotates vectors 90 degrees counter-clockwise around the x axis. Then find a matrix R_z that rotates vectors 90 degrees counter-clockwise around the z axis.

Since $R_x(e_1) = e_1$, $R_x(e_2) = e_3$ and $R_x(e_3) = -e_2$, $R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$.

Similarly, $R_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. We also have $R_z^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

b) Compute $A = R_z^{-1} R_x R_z$. Describe what this matrix does to an arbitrary vector.

$R_z^{-1} R_x R_z = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ which is a clockwise rotation by 90 degrees around the y-axis. In general, to rotate by an angle $-\theta$ about the y axis, first apply R_z, then rotate by θ about the x axis, then apply R_z^{-1}. Likewise, you can get an *arbitrary* rotation by utilizing just two motors in the right sequence. That's also how you pilot an airplane. With the stick you can point the nose of the plane up or down, or you can tilt the wings, but you can't directly turn left or right. To turn left or right, a pilot tilts the wings, then pulls "up", then untilts the wings.

10. Consider the matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

a) Is A diagonalizable? Why or why not?

Since A is symmetric, A is diagonalizable. (You can also work part (b) and see that GM=AM for each eigenvalue, but using symmetry is easier.)

b) Find the eigenvalues of A. For each eigenvalue, find a basis for the eigenspace.

The eigenvalues are 2 and -1. E_2 has dimension 1, with basis $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, while E_{-1} has dimension 2, with one possible basis being $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right\}$.

Another possible basis is $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

c) Compute A^{10}.

Using the first basis for E_{-1} we get

$$A^{10} = S\Lambda^{10}S^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1024 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ -1/2 & 1/2 & 0 \\ 1/6 & 1/6 & -1/3 \end{pmatrix} = \begin{pmatrix} 342 & 341 & 341 \\ 341 & 342 & 341 \\ 341 & 341 & 342 \end{pmatrix}.$$

Using the second basis we get

$$A^{10} = S\Lambda^{10}S^{-1} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1024 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix} = \begin{pmatrix} 342 & 341 & 341 \\ 341 & 342 & 341 \\ 341 & 341 & 342 \end{pmatrix}.$$

The details of the calculation are different, but the answer is the same.