Move ~~ado~~ a dot around

position = $\gamma(t)$

Parametrized curve.

E.g. $\gamma_1(t) = (t,0)$ $t = 2s$

$\gamma_2(t) = (2t, 0)$ $\gamma_2(s) = (2s, 0)$

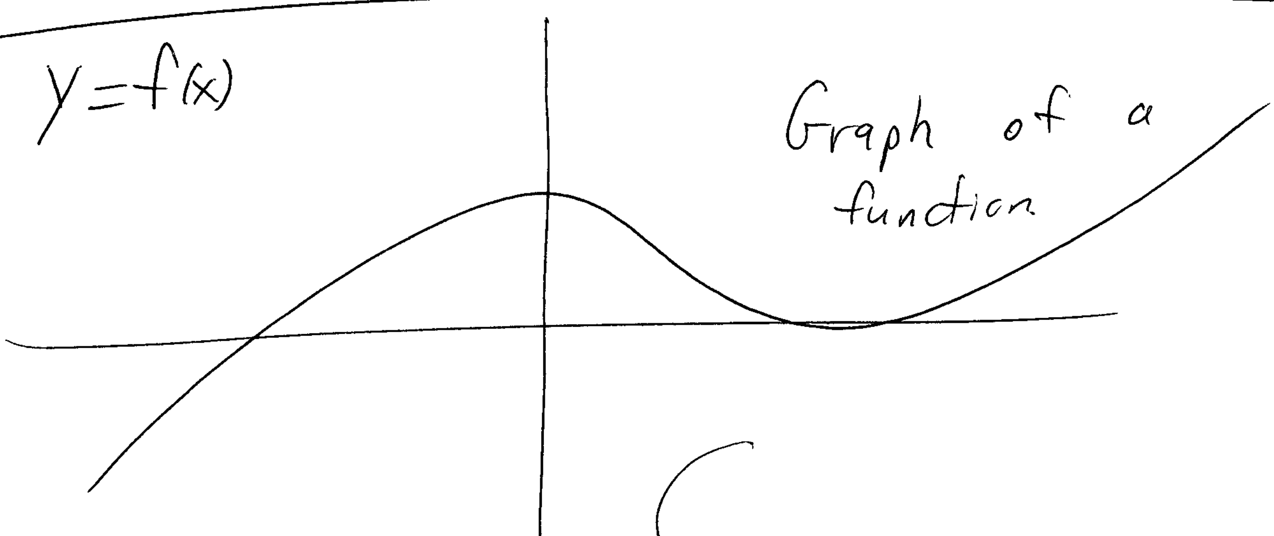
$\cancel{\gamma_3(t) = (t^3, 0)}$

$\gamma_3(u) = (u^3, 0)$

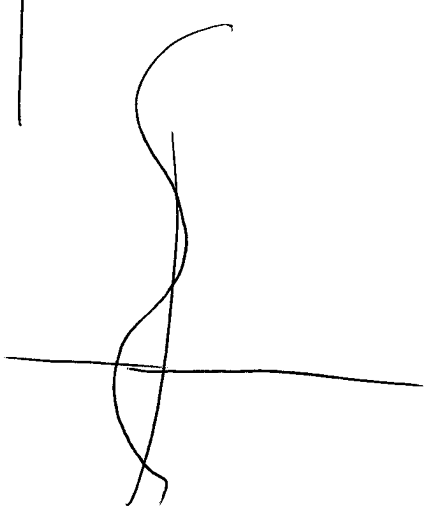
$t = u^3$

$u = \sqrt[3]{t}$

$y = f(x)$



$x = g(y)$



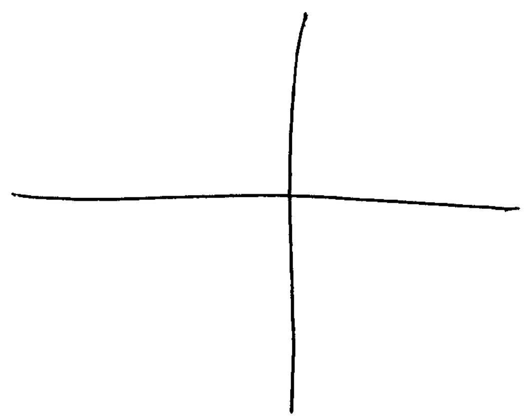
$F(x,y)=c$ Level curve.

$$\underbrace{y-f(x)}_{F(x,y)}=0$$

$\nabla F = (-f', 1)$

$F(x,y)=xy$

$F(x,y)=0$

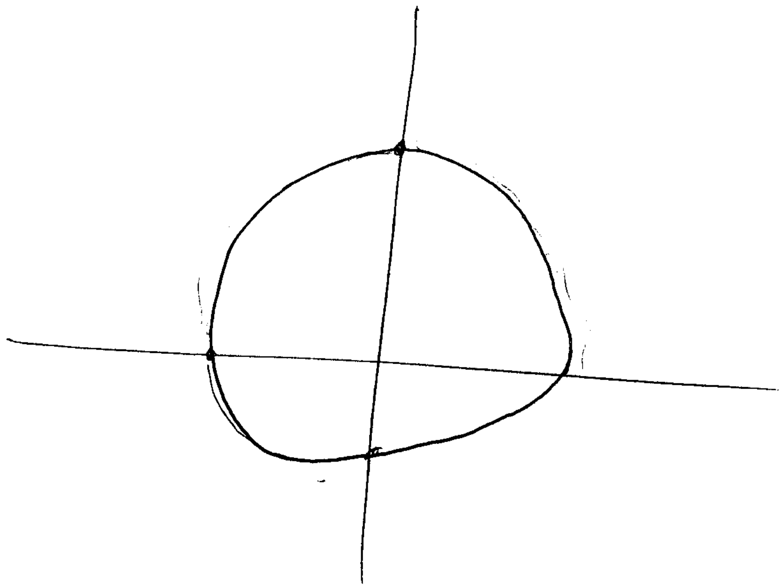


<u>Thm</u> If F is smooth and $\nabla F(x_0,y_0)\neq 0$, and $F(x_0,y_0)=c$, then curve $F(x,y)=c$ is smooth near (x_0,y_0). (is either a graph $y=f(x)$ w/ f smooth or $x=g(y)$ with g smooth)

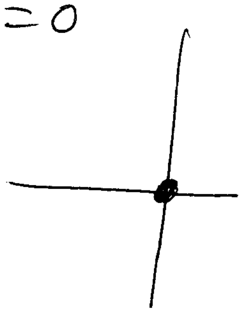
Implicit Function Thm.

Ex: $F(x,y) = x^2+y^2$ $\nabla F = (2x, 2y)$

$F(x,y)=1$



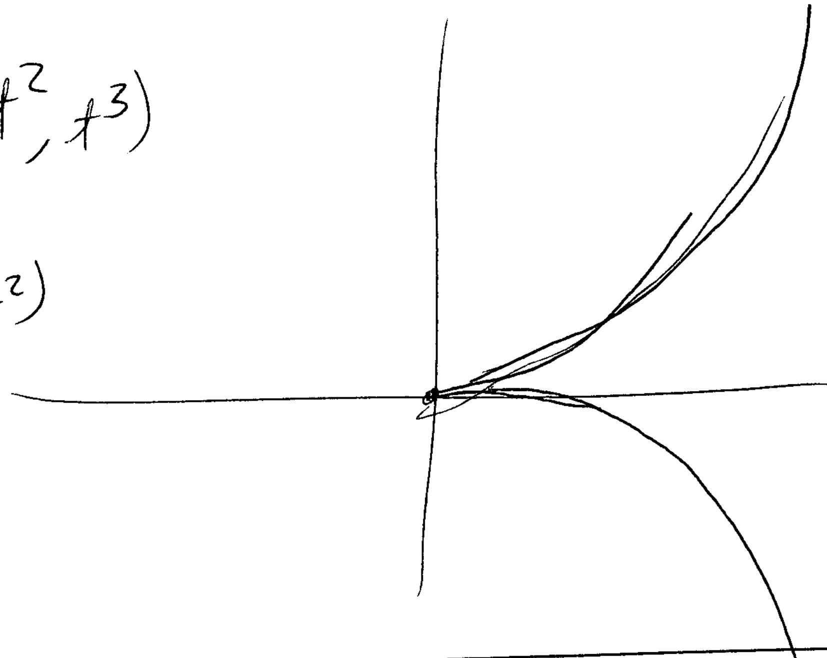
$F(x,y)=0$



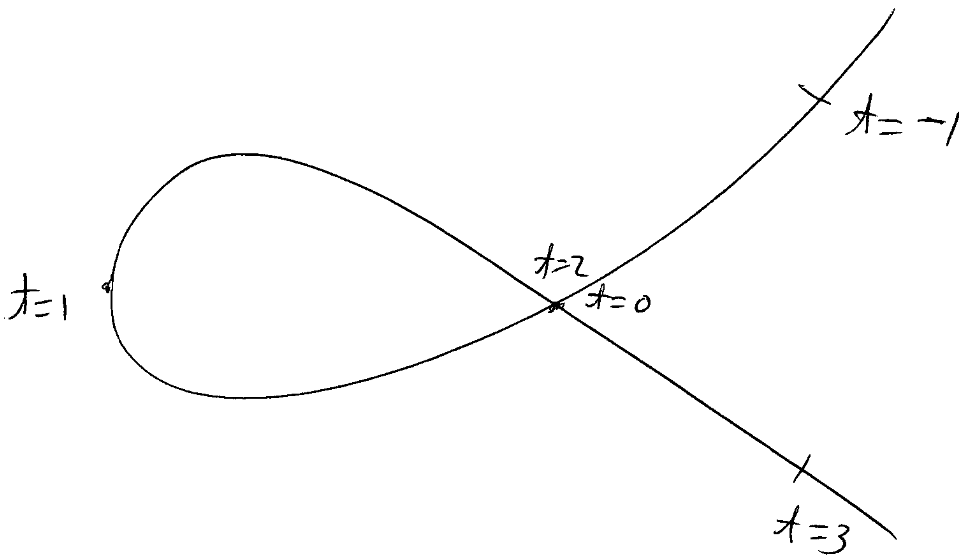
$\nabla F(0,0) = (0,0)$, Imp FT doesn't apply.

$$\gamma(t) = (t^2, t^3)$$

$$\mathring{\gamma}(t) = (2t, 3t^2)$$



If $\gamma(t)$ is a smooth function of t, and if $|\mathring{\gamma}(t_0)| \neq 0$, then, near $t = t_0$, ~~either~~ path of γ is a smooth graph.



Smooth if you think of "intersection" as 2 pts.

$\dot{\gamma}(t) =$ velocity.

$$\frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|} = \text{unit vector} = \vec{T} = \text{unit tangent vector.}$$

Useful to parametrize things so that $|\dot{\gamma}| = 1$.

$$\text{Arclength} = \int \text{speed} = \int |\dot{\gamma}(t)|\, dt = s$$

$$\vec{T} = \frac{d\gamma}{ds}$$

Contraction mapping principle.

If you have a map $T: \mathbb{R} \to \mathbb{R}$, and

$|T(x)-T(y)| \le \frac{1}{2}|x-y|$, then $\exists!$ pt

$\tilde{x}_0$ with $T(\tilde{x}_0)=\tilde{x}_0$.

Uniqueness: If $T(x_1)=x_1$ and $T(x_2)=x_2$,

$|T(x_1)-T(x_2)| = |x_1-x_2| \le \frac{1}{2}|x_1-x_2| \Rightarrow |x_1-x_2|=0$,

$\Rightarrow x_1=x_2$.

Existence.

Pick arbitrary x_0. Let $x_1=T(x_0)$, $x_2=T(x_1)$

$x_3=T(x_2)$, etc.

Claim: $x_n \to x_\infty$

$$x_n = (x_n-x_{n-1}) + (x_{n-1}-x_{n-2}) + (x_{n-2}-x_{n-3}) + \cdots + (x_1-x_0) + x_0$$

$$|x_n-x_{n-1}| = |T(x_{n-1})-T(x_{n-2})| \le \frac{1}{2}|x_{n-1}-x_{n-2}|$$

$$T(x_\infty) = T(\lim x_n) = \lim T(x_n) = \lim x_{n+1} = x_\infty$$

Inverse Function Thm

Let $f: \mathbb{R} \to \mathbb{R}$ be a smooth function with $f(x_0) = y_0$. If $f'(x_0) \neq 0$, then, restricting our domain to ~~an~~ a neighborhood of x_0, there is a smooth inverse function g.

That is, for all $y \approx y_0$, $\exists!\, x \approx x_0$ with $f(x) = y$, namely $x = g(y)$.

Furthermore, ~~$g'(y)$~~ $\frac{dg}{dy} = \frac{1}{df/dx}$

Pf WLOG, assume $x_0 = 0 = y_0$ and $f'(0) = 1$

$\lim\limits_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 1$ $\qquad$ $\frac{f(x)}{x} \Rightarrow 1$ as $x \to 0$

$f(x) = x + \text{error}$ with $\frac{\text{error}}{x} \to 0$

~~Assume~~ Pick nbhd with $|\text{error}| < \frac{x}{2}$, $|f' - 1| < \frac{1}{2}$

~~Solve~~ Solve $f(x) = y$ w/ y fixed.

$$T(x) = x + y - f(x)$$

$< \frac{|x_2 - x_1|}{2}$

$$T(x_1) - T(x_2) = x_1 + y - f(x_1) - x_2 - y + f(x_2)$$
$$= (f(x_2) - f(x_1)) - (x_2 - x_1) = \int_{x_1}^{x_2} (f'(x) - 1)\, dx$$

$$Y_1 - Y_2 = f'(x_0)(X_1 - X_2) + \text{error}$$

$$(X_1 - X_2) = \frac{1}{f'(x_0)}(Y_1 - Y_2) + \text{error.}$$

$$g'(y_0) = \frac{1}{f'(x_0)}$$

$$\frac{dg}{dy}(y) = \frac{1}{f'(x)} = \frac{1}{f'(g(y))} \qquad x = g(y)$$

$$g''(y) = -\frac{\frac{d}{dy}\left(f'(g(y))\right)}{\left(f'(g(y))\right)^2} = -\frac{f''(g(y)) \cdot g'(y)}{\left(f'(g(y))\right)^2}$$

~~If $\dot{\gamma}(t_0) \neq 0$, either~~ $\left.\frac{dx}{dt}\right|_{t_0}$ $\cancel{\neq 0}$ or

$\gamma(t) = (\gamma_1(t), \gamma_2(t))$

$x = \gamma_1(t)$

$y = \gamma_2(t)$

If $\dot{\gamma}(t_0) \neq 0$, either $\dot{\gamma}_1(t_0) \neq 0$ or $\dot{\gamma}_2(t_0) \neq 0$.

If $\dot{\gamma}_1(t_0) \neq 0$, then $t = g(x)$ for smooth g.

($g = \gamma_1^{-1}$ exists by inverse function thm)

$y = \gamma_2(g(x)) =$ smooth function of x.

If $\dot{\gamma}_2(t_0) \neq 0$, $x = \gamma_1(\tilde{g}(y))$, where $t = \tilde{g}(y)$

$\gamma(t)$

velocity $= \dot{\gamma}(t)$

Speed $= |\dot{\gamma}(t)| = \sqrt{\dot{\gamma}(t) \cdot \dot{\gamma}(t)}$

is smooth as long as $\dot{\gamma} \neq 0$.

arclength $= s(t) = \int (\text{Speed})\, dt$

$=$ Smooth function of t

$\frac{ds}{dt} =$ Speed. $=$ smooth

By inverse function thm, $t = \phi(s)$

$\gamma(t) = \gamma(\phi(s)) = \tilde{\gamma}(s)$

Def. A smooth parametrized curve is <u>regular</u> if $|\dot{\gamma}| \neq 0$