Reparametrization

$\gamma : (a,b) \to \mathbb{R}^3$ Smooth curve,

$a < t < b$

$\tilde{\gamma} : (\tilde{a}, \tilde{b}) \to \mathbb{R}^3$

$\tilde{a} < \tilde{t} < \tilde{b}$

$t = \phi(\tilde{t})$

$\tilde{t} = \phi^{-1}(t)$

$\phi : (\tilde{a}, \tilde{b}) \to (a,b)$ smooth, ϕ^{-1} smooth

$\gamma(t) = \tilde{\gamma}(\tilde{t})$

Ex: $\gamma(t) = (\cos(t), \sin(t))$

$\tilde{\gamma}(\tilde{t}) = (\sin(\tilde{t}), \cos(\tilde{t}))$

$t = \frac{\pi}{2} - \tilde{t}$

$= \phi(\tilde{t})$

Ex: $\gamma(t) = (t, t)$

$\tilde{\gamma}(\tilde{t}) = (\tilde{t}^3, \tilde{t}^3)$

$\phi(\tilde{t}) = \tilde{t}^3 = t$

$\phi^{-1}(t) = t^{1/3}$

Reparametrization of regular curve is regular.

If a ~~curva~~ curve is regular, it has a unit-speed reparametrization.

$$s(t) = \int \text{speed}\, dt = \int \sqrt{(\dot{\gamma}\cdot\dot{\gamma})}\, dt$$

$$= \int \sqrt{\left(\frac{d\gamma_1}{dt}\right)^2 + \left(\frac{d\gamma_2}{dt}\right)^2 + \left(\frac{d\gamma_3}{dt}\right)^2}\, dt$$

$=$ smooth function of t

$$\frac{ds}{dt} = \text{speed} = |\dot{\gamma}| \neq 0$$

$t = \phi(s)$ for ϕ smooth, ϕ^{-1} smooth.

define.

$$\tilde{\gamma}(s) = \gamma(t) = \gamma(\phi(s))$$

$$\left|\frac{d\tilde{\gamma}}{ds}\right| = \left|\frac{d\gamma}{dt}\frac{dt}{ds}\right| = |\dot{\gamma}|/|\dot{\gamma}| = 1$$

$\gamma(t) = (t, \cosh(t))$

Digression.

$\cosh(t) = \dfrac{e^{t}+e^{-t}}{2}$

$\cos(t) = \dfrac{e^{it}+e^{-it}}{2}$

$\sinh(t) = \dfrac{e^{t}-e^{-t}}{2}$

$\sin(t) = \dfrac{e^{it}-e^{-it}}{2i}$

$\dfrac{d\sinh(t)}{dt} = \dfrac{e^{t}+e^{-t}}{2} = \cosh(t)$

$\dfrac{d\sin(t)}{dt} = \cos(t)$

$\dfrac{d\cosh(t)}{dt} = \dfrac{e^{t}-e^{-t}}{2} = \sinh(t)$

$\dfrac{d\cos(t)}{dt} = \boxed{-}\sin t$

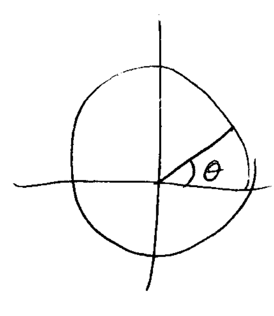
$\dfrac{d^2f}{dt^2} = f$

$\dfrac{d^2f}{dt^2} = -f$

$\dfrac{d^2f}{dt^2} = cf$

$\cos^2 + \sin^2 = 1$

$\cosh^2 - \sinh^2 = 1$



$\gamma(t) = (t, \cosh(t))$

$\dot{\gamma} = (1, \sinh(t))$

$|\dot{\gamma}| = \sqrt{1^2 + \sinh^2(t)} = \cosh(t)$

$S = \int |\dot{\gamma}| dt = \sinh(t)$

We say a curve ~~ha~~ is periodic w/ period T if, $\forall t$, $\gamma(t+T) = \gamma(t)$

For regular closed curves, ~~can't~~ can use unit-speed parametrization, so $T =$ length.

"The period" = smallest positive period.

A periodic curve has a self-intersection if $\gamma(a) = \gamma(b)$ for some $a, b \in \mathbb{R}$ with $b - a$ not a period.

1D Inverse function thm:

If $f: \mathbb{R} \to \mathbb{R}$ is smooth, and if $f'(x_0) \neq 0$, then locally f has a smooth inverse.

and $\frac{d(f^{-1})}{d\cancel{x}y} = \frac{1}{df/dx}$ $\left(\frac{dx}{dy} = \frac{1}{dy/dx} \right)$

2D Inverse function thm:

If $f: \mathbb{R}^2 \to \mathbb{R}^2$ is smooth, and if

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}$$ is invertible at (x_0, y_0)

then locally f has a smooth inverse

and $D(f^{-1}) = (Df)^{-1}$

Implicit function thm (2D)

Let $F: \mathbb{R}^2 \to \mathbb{R}$ be a smooth function,

If $\nabla F(x_0, y_0) \neq \vec{0}$ and $F(x_0, y_0) = c$, then the level curve ~~curv set~~ $F(x,y) = c$ is smooth near (x_0, y_0).

Specifically, If $\frac{\partial F}{\partial x}(x_0, y_0) \neq 0$, we can write $x = g(y)$ and if $\frac{\partial F}{\partial y} \neq 0$ we can write $y = f(x)$

Pf: ~~Let $H(x,y) =$~~ Suppose $\frac{\partial F}{\partial y} \neq 0$

Let $H(x,y) = (x, F(x,y))$

$$DH = \begin{pmatrix} 1 & 0 \\ \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{pmatrix} \qquad \det(DH) = \frac{\partial F}{\partial y} \neq 0$$

So H has a smooth inverse.

$$(x,y) = H^{-1}(x, F(x,y))$$

On level set

$$(x,y) = H^{-1}(x, c)$$

$y =$ 2nd entry in $H^{-1}(x,c)$
$=$ smooth function of x

Ex: $F(x,y) = x^2+y^2$ $\qquad c=1$

$\nabla F = (2x, 2y)$