A smooth paramet~~t~~rized curve is a ~~m~~ (smooth) map $\gamma : (a,b) \longrightarrow \mathbb{R}^3$

It is <u>regular</u> if $\dot{\gamma} \neq 0$

Locally, image of regular curve in $\mathbb{R}^2$ $\iff$ graph of a smooth function

$\iff$ level set of function w/ non-vanishing gradient

A curve is regular $\iff$ it has a unit-speed (re)~~p~~paremetrization.

If $\vec{v}(t)$ and $\vec{w}(t)$ are smooth vector-valued functions,
then

$$\frac{d}{dt}(\vec{V}\cdot\vec{w}) = (\dot{\vec{V}}\cdot\vec{w}) + (\vec{V}\cdot\dot{\vec{w}})$$

$$\frac{d}{dt}(\vec{V}\cdot\vec{w}) = \frac{d}{dt}\sum_i V_i W_i = \sum_i \dot{V}_i W_i + V_i \dot{W}_i$$

$$= \dot{\vec{V}}\cdot\vec{W} + \vec{V}\cdot\dot{\vec{W}}$$

Cor: If $\vec{V}\cdot\vec{W} =$ constant, $\dot{\vec{V}}\cdot W + \vec{V}\cdot\dot{\vec{W}} = 0$

Cor: If $|\vec{V}| =$ constant, $\dot{\vec{V}}\cdot\vec{V} = 0$.

Imagine unit-speed curve $\gamma(t)$.

$\vec{T} = \dot{\gamma} =$ unit vector $=$ unit tangent vector.

$\ddot{\gamma} \perp \dot{\gamma}$

<u>Def</u>: <u>Curvature</u> of γ is $\kappa = |\ddot{\gamma}|$ for unit-speed curve. $= |d\vec{T}/ds|$

<u>Def</u>: Curvature of any curve is $\kappa = |d\vec{T}/ds|$
where $s =$ arclength.

<u>Thm</u> If $K=0$ always, straight line.

Pf: Parametrize by arclength.

$$\ddot{\gamma}=0, \quad \dot{\gamma} = T = \text{constant}$$

$$\gamma(s) = \vec{T}s + \text{constant} = \text{line.}$$

<u>Ex:</u> Circle of radius R.

$$\gamma(t) = (R\cos t, R\sin t) \qquad \frac{d\gamma}{dt} = (-R\sin t, R\cos t)$$

$$\left|\frac{\partial \gamma}{\partial t}\right| = R \qquad s = Rt$$

$$t = s/R$$

$$\tilde{\gamma}(s) = \left(R\cos\left(\frac{s}{R}\right), R\sin\left(\frac{s}{R}\right)\right)$$

$$\vec{T} = \left(-\sin\left(\frac{s}{R}\right), \cos\left(\frac{s}{R}\right)\right)$$

$$\frac{d\vec{T}}{ds} = \left(-\frac{1}{R}\cos\left(\frac{s}{R}\right), -\frac{1}{R}\sin\left(\frac{s}{R}\right)\right)$$

$$K = \left|\frac{d\vec{T}}{ds}\right| = \frac{1}{R}$$

Thm

For any (regular, smooth) curve $\gamma(t)$,

$$\kappa(t) = \frac{|\dot\gamma \times \ddot\gamma|}{|\dot\gamma|^3}$$

$$= \frac{|\ddot\gamma| \sin\theta}{|\dot\gamma|^2}$$

$\gamma =$ length

$\dot\gamma =$ length / time

$\ddot\gamma =$ length / time^2

$\dot\gamma \times \ddot\gamma =$ length^2 / time^3

$|\dot\gamma|^3 =$ length^3 / time^3

$\kappa =$ 1 / length.

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pf: $$\vec T = \frac{\dot\gamma}{|\dot\gamma|} = \dot\gamma\,(\dot\gamma\cdot\dot\gamma)^{-1/2}$$

$$\frac{d\vec T}{dt} = \ddot\gamma\,(\dot\gamma\cdot\dot\gamma)^{-1/2} - \dot\gamma\,(\dot\gamma\cdot\ddot\gamma)(\dot\gamma\cdot\dot\gamma)^{-3/2}$$

$$= \frac{\ddot\gamma\,(\dot\gamma\cdot\dot\gamma) - \dot\gamma\,(\dot\gamma\cdot\ddot\gamma)}{|\dot\gamma|^3} = \frac{-\dot\gamma \times (\dot\gamma \times \ddot\gamma)}{|\dot\gamma|^3}$$

$$\kappa = |dT/ds| = \frac{|dT/dt|}{|ds/dt|} = \frac{|\dot\gamma \times \ddot\gamma| / |\dot\gamma|^2}{|\dot\gamma|}$$

$$= \frac{|\dot\gamma \times \ddot\gamma|}{|\dot\gamma|^3} \qquad \blacksquare$$

$$\frac{d}{ds} = \frac{d/dt}{ds/dt} = \frac{d/dt}{|\dot\gamma|}$$

Helix: $\gamma(t) = (a\cos(t), a\sin(t), bt)$

$$\dot{\gamma} = (-a\sin(t), \; a\cos(t), \; b)$$

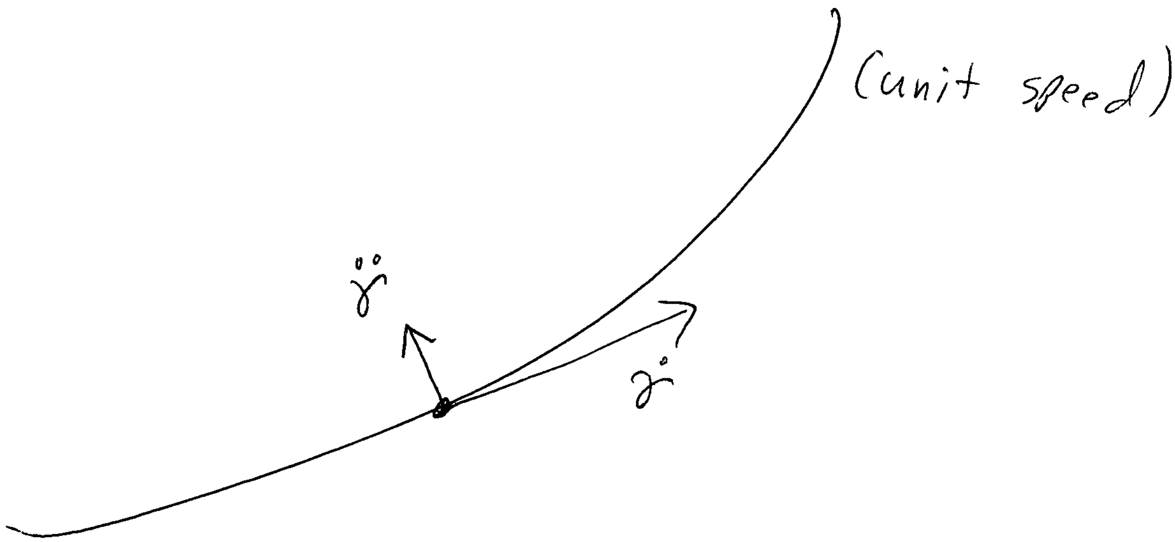
$$\ddot{\gamma} = (-a\cos(t), -a\sin(t))$$

$$\dot{\gamma} \times \ddot{\gamma} = \begin{vmatrix} i & j & k \\ -a\sin & a\cos & b \\ -a\cos & -a\sin & 0 \end{vmatrix} = (ab\sin(t), -ab\cos(t), a^2)$$

$$|\dot{\gamma} \times \ddot{\gamma}| = \sqrt{a^2b^2 + a^4} = a\sqrt{a^2+b^2}$$

$$|\dot{\gamma}| = \sqrt{a^2+b^2}$$

$$k = \frac{|\dot{\gamma} \times \ddot{\gamma}|}{|\dot{\gamma}|^3} = \frac{a\sqrt{a^2+b^2}}{(\sqrt{a^2+b^2})^3} = \frac{a}{a^2+b^2}$$



$|\ddot{\gamma}| = \kappa$

$\vec{N}$ = principal normal = $\dfrac{\ddot{\gamma}}{\kappa}$

$\vec{B}$ = binormal = $\vec{T} \times \vec{N}$

$\ddot{\gamma} = \kappa \vec{N}$

First order: γ = line in $\vec{T}$ direction

2nd order γ = ~~ca~~ circle of radius $\frac{1}{\kappa}$ in $(\vec{T}, \vec{N})$ plane.

3rd order ($\dddot{\gamma}$) tells

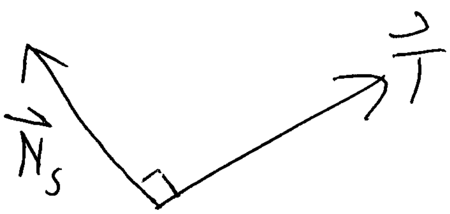
(a) changes in curvature

(b) changes in plane (torsion)

Curves in $\mathbb{R}^2 \subset \mathbb{R}^3$

$(x,y) = (x,y,0)$

$\vec{N}_s$ = signed normal vector = $90°$ ccw rotation of $\vec{T}$.



Parametrize by arclength.

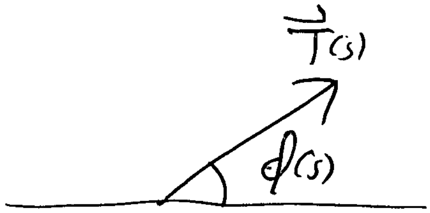
$$\dot{\vec{T}} = a\vec{T} + b\vec{N}_s$$

$$0 = \vec{T}\cdot\dot{\vec{T}} = a \Rightarrow \dot{\vec{T}} = b\vec{N}_s$$

let $\kappa_s = b$ = signed curvature.

$$\vec{N} = \pm\vec{N}_s$$

$$\kappa = |\kappa_s|$$



$$\vec{T}(s) = \left(\cos(\phi(s)),\ \sin(\phi(s))\right)$$

ϕ $$\vec{N}_s(s) = \left(-\sin(\phi(s)),\ \cos(\phi(s))\right)$$

$$\frac{d\vec{T}}{ds} = \dot{\phi}(s)\left(-\sin(\phi),\ \cos(\phi)\right) = \dot{\phi}(s)\,\vec{N}_s$$

$$K_s = \frac{d\phi}{ds} = \frac{d\phi/dt}{ds/dt}$$

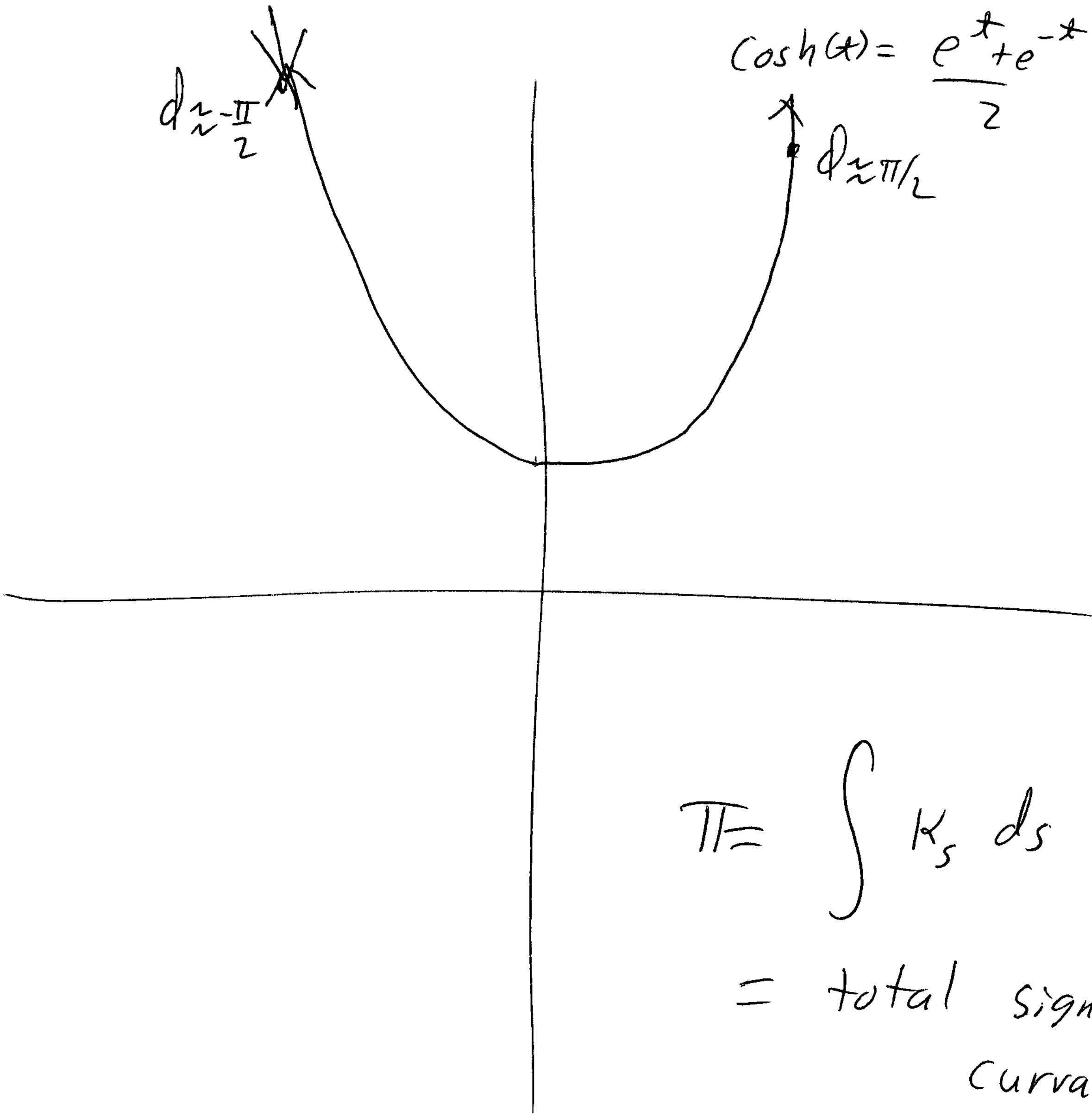
Ex: $\gamma(t) = (t,\ \cosh(t))$

$$\dot{\gamma} = (1,\ \sinh(t)) \qquad |\dot{\gamma}| = \sqrt{1+\sinh^2} = \cosh(t)$$

$$\phi = \cancel{\sin}\,\tan^{-1}(\sinh(t))$$

$$\frac{d\phi}{dt} = \frac{\cosh(t)}{1+\sinh^2(t)} = \frac{1}{\cosh(t)}$$

$$\frac{d\phi}{ds} = \frac{1}{\cosh^2 t}$$



$$\pi = \int \kappa_s \, ds$$

$=$ total signed curvature.

For any curve, $\int \kappa_s \, ds = \int \frac{d\phi}{ds} \, ds$

$$= \Delta \phi$$

For <u>closed</u> curve, $\int_0^\ell \kappa_s \, ds = 2\pi n$.