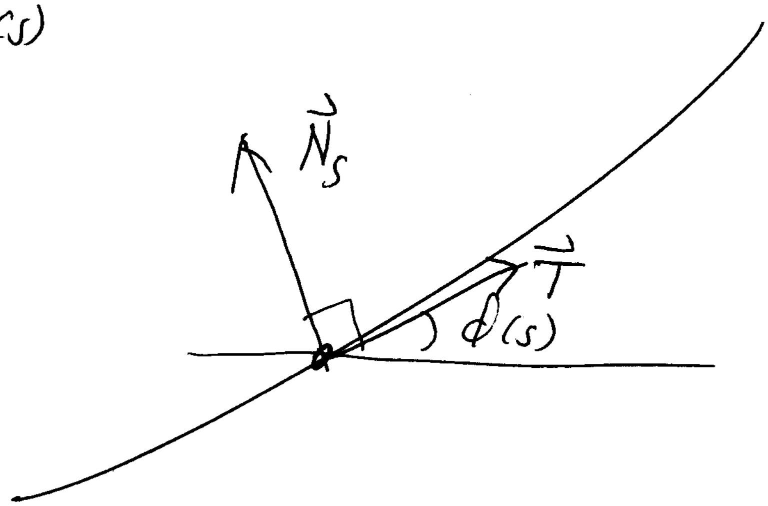
Plane curve $\vec{\gamma}(s)$ (unit-speed)

$\vec{T} = \frac{d\vec{\gamma}}{ds}$

$\vec{N}_s = 90^\circ$ ccw rotation of $\vec{T}$

$\dot{\vec{T}}(s) = \kappa_s(s)\vec{N}_s(s)$



$\kappa_s(s) = \frac{d\phi}{ds}$

For closed curve, $\int_0^{\ell} \kappa_s(s)\,ds = 2\pi n.$

A *direct isometry* is a rotation followed by a translation

An *inverse isometry* is a rotation followed by a translation followed by a reflection

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Applying a *direct* isometry to $\gamma(s)$ doesn't change $k_s(s)$.

Inverse isometry changes $k_s(s)$ to $-k(s)$

Thm Given any smooth function $\cancel{k_s(s)}$, $f(s)$ there is a curve whose signed curvature is $k_s(s) = f(s)$. This curve is unique up to direct isometry.

Pf (existence). Let $\phi(s) = \int_0^s f(s')\,ds'$

Let $\vec{T}(s) = (\cos(\phi(s)), \sin(\phi(s)))$

Let $\vec{\gamma}(s) = \int \vec{T}(s)\,ds$ with $\vec{\gamma}(0) = \vec{0}$

<u>Uniqueness</u> Suppose $\gamma_1(0) = \gamma_2(0) = 0$

$\dot{\gamma}_1(0) = \dot{\gamma}_2(0) = (1,0)$

and $K_{s,1}(s) = K_{s,2}(s)$

Then $\frac{d}{ds}(\phi_1(s) - \phi_2(s)) = 0$, so $\phi_1(s) = \phi_2(s)$

So $\vec{T}_1 = \vec{T}_2$, so $\frac{d\vec{\gamma}_1}{ds} = \frac{d\vec{\gamma}_2}{ds}$, so

$\vec{\gamma}_1 - \vec{\gamma}_2 = \text{constant} = 0$, so $\vec{\gamma}_1(s) = \vec{\gamma}_2(s)$

Thm $\kappa(s)$ determines γ up to isometry

$\boxed{\text{if } \kappa(s) \text{ is never } 0}$

Pf $\kappa_s = \pm\kappa$, sign never changes since κ_s is continuous and $\kappa \neq 0$, so you know κ_s up to overall sign, so you know γ up to direct isometry + possible reflection.

Counter example w/ $\kappa = 0$

Curve is 3D space.

$\vec{\gamma}(s)$ – unit speed.

$$\vec{T} = \frac{d\vec{\gamma}(s)}{ds}$$

$$\frac{d\vec{T}}{ds} = \kappa \vec{N}$$

$(\vec{T}, \vec{N})$ span a plane called the tangent plane.

$\vec{B} = \vec{T} \times \vec{N}$ is vector $\perp$ to tangent plane

$$\left.\begin{aligned} \vec{T} &= \text{tangent} \\ \vec{N} &= \text{Principal normal} \\ \vec{B} &= \text{binormal.} \end{aligned}\right\} \begin{array}{c} \cancel{\text{Frenet}} \\ \text{Frenet} \\ \text{moving} \\ \text{frame} \end{array} \qquad \begin{aligned} &\vec{B} \perp \vec{N} \\ &\vec{B} \perp \vec{T} \\ &\vec{N} \perp \vec{T}, \end{aligned} \qquad |B| = |N| = |T| = 1$$

If $\vec{V} \cdot \vec{V} = \text{const}$, $\quad \dot{\vec{V}} \cdot \vec{V} = 0$

$\vec{V} \cdot \vec{W} = \text{const} \quad \dot{V} \cdot W + V \cdot \dot{W} = 0$

$$\dot{\vec{B}} \cdot \vec{B} = 0$$

$$\dot{\vec{B}} \cdot \vec{T} = -\vec{B} \cdot \dot{\vec{T}} = -\vec{B} \cdot \kappa \vec{N} = 0$$

$\dot{\vec{B}} = \text{multiple of } \vec{N} = -\tau \vec{N} \qquad \tau = \text{torsion.}$

$$\dot{\vec{N}} \cdot \vec{T} = -\vec{N} \cdot \dot{\vec{T}} = -\kappa$$

$$\dot{\vec{N}} \cdot \vec{N} = 0$$

$$\dot{\vec{N}} \cdot \vec{B} = -\vec{N} \cdot \dot{\vec{B}} = \tau$$

$$\frac{d}{dt} \begin{pmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{pmatrix}$$

Frenet - Serret equations

Echoes to 2D

$$\frac{d}{dt} \begin{pmatrix} \vec{T} \\ \vec{N}_s \end{pmatrix} = \begin{pmatrix} 0 & \kappa_s \\ -\kappa_s & 0 \end{pmatrix} \begin{pmatrix} \vec{T} \\ \vec{N}_s \end{pmatrix}$$

In 3D, κ and τ determine a curve up to direct isometry. as long as $\kappa \neq 0$.