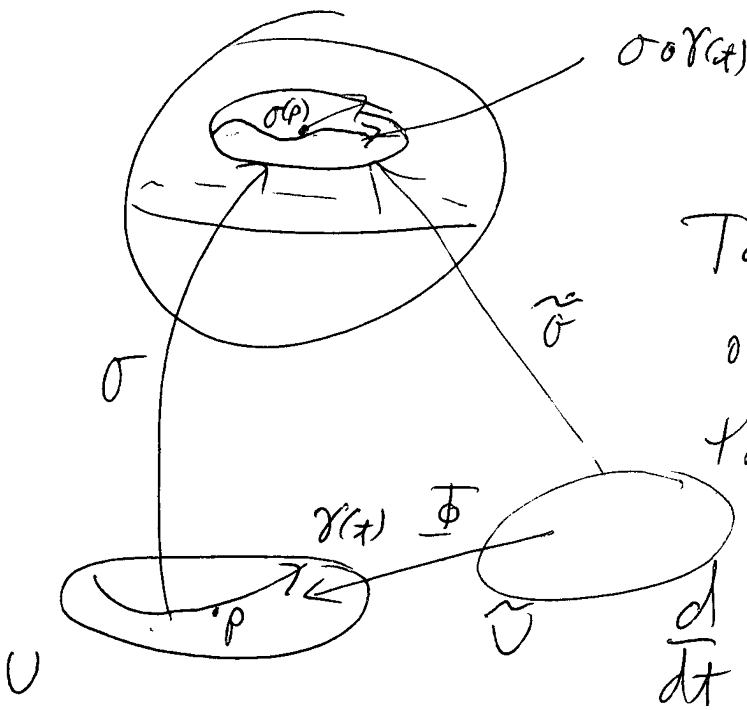


Tangent vector is velocity of a curve on S through $\sigma(p)$.

$$\frac{d}{dt}\left(\sigma \circ \gamma(t)\right) = \sigma_u \frac{du}{dt} + \sigma_v \frac{dv}{dt}$$

$$\in \operatorname{Span}\{\sigma_u, \sigma_v\}.$$

$$\gamma(t) = (u(t), v(t))$$

$$T_{\sigma(p)} S = \operatorname{Span}\{\sigma_u, \sigma_v\}.$$

$$(u,v) = \Phi(\tilde{u}, \tilde{v})$$

$$\tilde{\sigma}(\tilde{u}, \tilde{v}) = \sigma \circ \Phi(\tilde{u}, \tilde{v})$$

$$\tilde{\sigma}_{\tilde{u}} = \sigma_u \frac{\partial u}{\partial \tilde{u}} + \sigma_v \frac{\partial v}{\partial \tilde{u}}$$

$$\tilde{\sigma}_{\tilde{v}} = \sigma_u \frac{\partial u}{\partial \tilde{v}} + \sigma_v \frac{\partial v}{\partial \tilde{v}}$$

$T_{\sigma(p)} S$ depends only on S, not on σ.

$\sigma_u \times \sigma_v$ is $\perp$ $T_{\sigma(p)} S$. $\quad N_\sigma = \dfrac{\sigma_u \times \sigma_v}{|\sigma_u \times \sigma_v|}$

Level Surfaces (digression)

$F(x,y,z) = c \qquad \nabla F \neq 0.$

$$N = \pm \frac{\nabla F}{|\nabla F|}$$

<u>Thm</u> If $\frac{\partial F}{\partial z} \neq 0$, can write $z = f(x,y)$.

$\sigma(u,v) = (u, v, f(u,v))$

$\sigma_u = (1, 0, \frac{\partial f}{\partial u})$

$\sigma_v = \cancel{(1,0} \; (0, 1, f_v)$

$\sigma_u \times \sigma_v = (-f_u, -f_v, 1)$

$\nabla F \perp \sigma_u$, $\cancel{F_u \neq f_u}$

$\nabla F \perp \sigma_v$

$\cancel{-f_u}$

$0 = F_x + \frac{\partial f}{\partial u} F_z \Rightarrow f_u = -\frac{F_x}{F_z}$

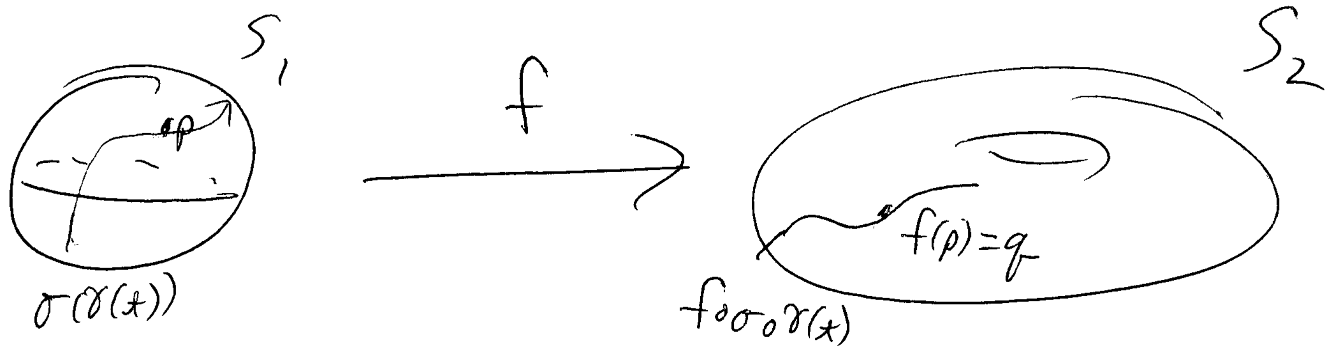
$0 = F_y + f_v F_z \Longrightarrow f_v = -F_y / F_z$

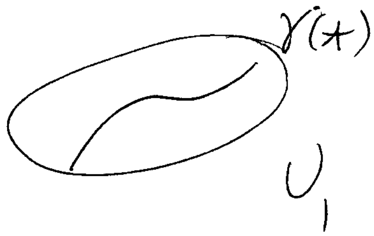
$\sigma_u = (1, 0, -F_x/F_z)$

$\sigma_v = (0, 1, -F_y/F_z)$

$\sigma_u \times \sigma_v = \left(+\frac{F_x}{F_z}, \frac{F_y}{F_z}, 1 \right)$

$= \nabla F / F_z$





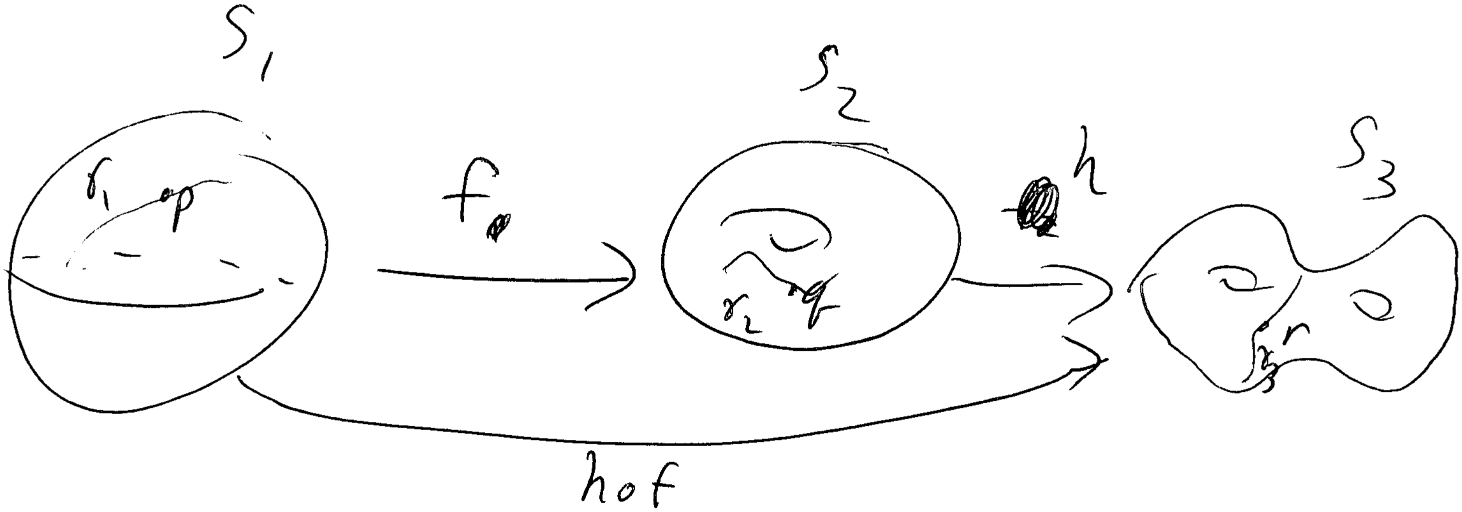
f sends paths on $S_1 \longrightarrow$ paths on S_2

velocities on $S_1 \longrightarrow$ velocities on S_2

$$Df: T_p S_1 \longrightarrow T_q S_2$$

Ex: $S_1 = S_2 = \mathbb{R}^2$

$$\dot{(f\circ\gamma)} = \begin{pmatrix} \frac{\partial f^1}{\partial u} & \frac{\partial f^1}{\partial v} \\ \frac{\partial f^2}{\partial u} & \frac{\partial f^2}{\partial v} \end{pmatrix} \begin{array}{c} \begin{pmatrix} \dot\gamma_1 \\ \dot\gamma_2 \end{pmatrix} \end{array} \begin{array}{l} \dot u \\ \dot v \end{array}$$

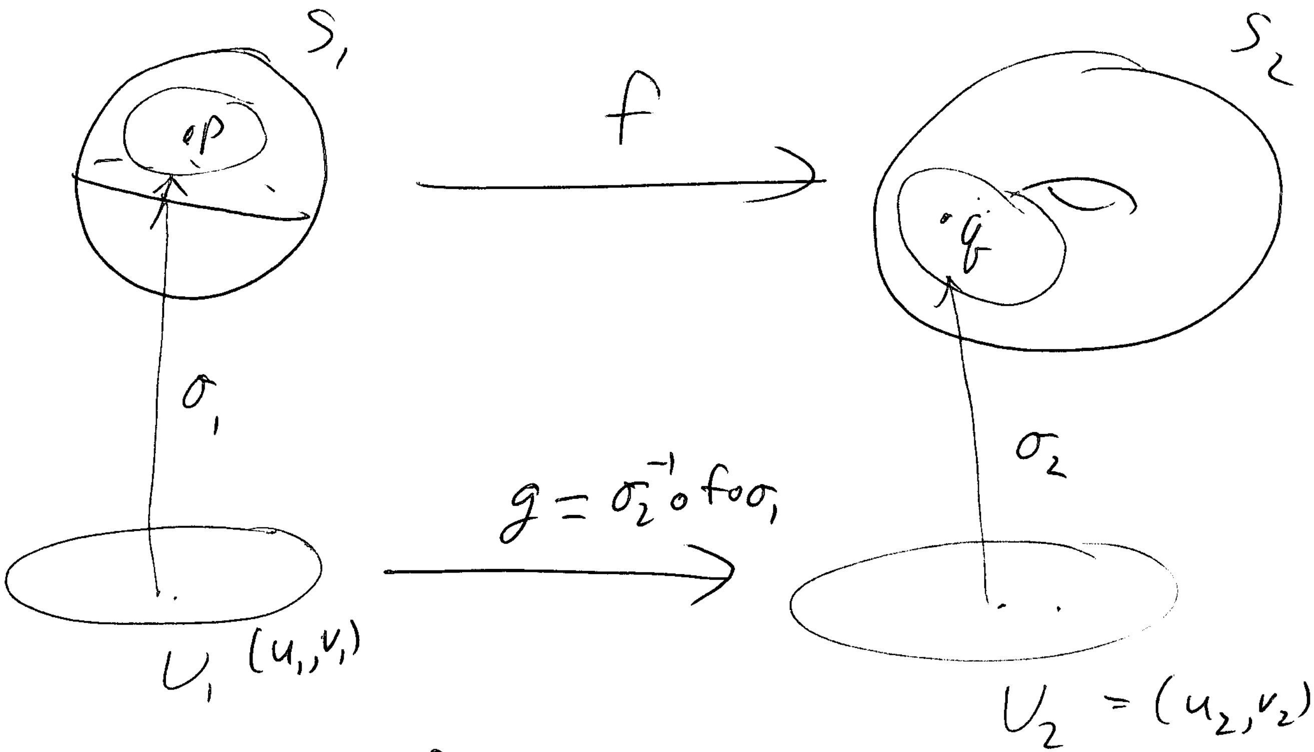


$q = f(p)$
$r = h(q)$

$$Df : T_p S_1 \to T_q S_2$$
$$Dh : T_q S_2 \to T_r S_3$$
$$D(h \circ f) : T_p(S_1) \to T_r(S_3)$$

<u>Thm</u> $D(h \circ f) = Dh \circ Df$

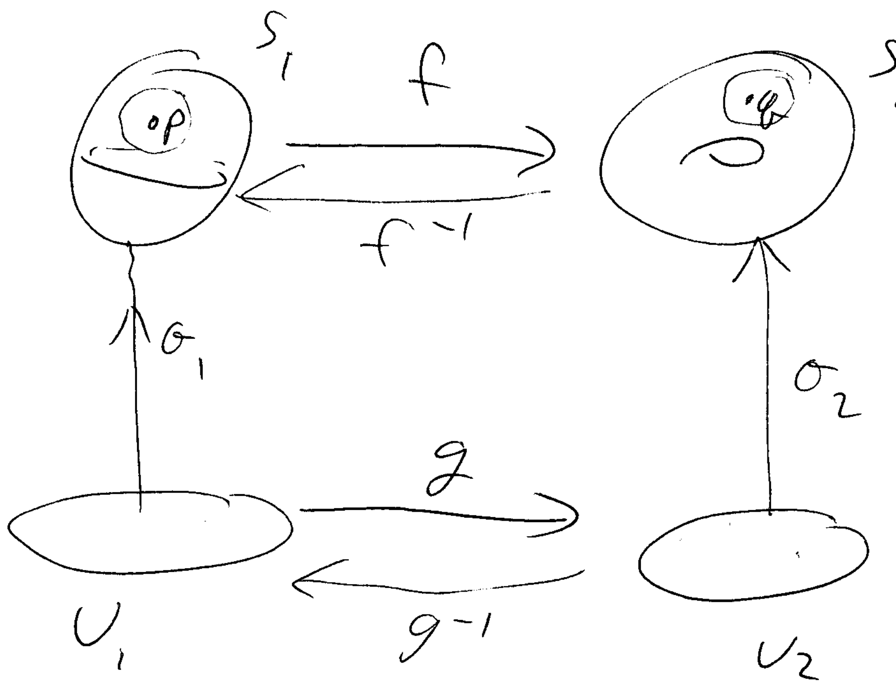
$\gamma_2(t) = f \circ \gamma_1(t)$ $\quad$ $\gamma_3(t) = h(\gamma_2(t))$
$$= (h \circ f)(\gamma_1(t))$$



$$Dg = \begin{pmatrix} \frac{\partial u_2}{\partial u_1} & \frac{\partial u_2}{\partial v_1} \\ \frac{\partial v_2}{\partial u_1} & \frac{\partial v_2}{\partial v_1} \end{pmatrix}$$

Thm Let $f: S_1 \to S_2$ be a smooth map of surfaces. Then f is a local diffeo if and only if Df is invertible.
($\Longleftrightarrow Dg$ is a nonsingular matrix)

Pf



By IFT on $\mathbb{R}^2$, g is a local diffeo if and only if Dg is non-singular.

Define $f^{-1} = \sigma_1 \circ g^{-1} \circ \sigma_2^{-1}$

Curves & Surfaces are subsets of $\mathbb{R}^3$ (or $\mathbb{R}^2$) that locally look like $\mathbb{R}$ or $\mathbb{R}^2$.

<u>Curves:</u> Parametrization : $\gamma: \mathbb{R} \to \mathbb{R}^3$

A <u>regular</u> curve has $\dot{\gamma} \neq 0$ always.

If γ is regular, reparametrize by arclength.

$\cdot = \frac{d}{dt}$; $' = \frac{d}{ds}$ $\qquad \frac{ds}{dt} = |\dot{\gamma}| =$ speed.

$$\vec{T} = \gamma' = \frac{\dot{\gamma}}{|\dot{\gamma}|}$$

$$\vec{T}' = \kappa \vec{N} \qquad \kappa = \text{curvature} = |\gamma''|$$

$$= \frac{|\dot{\gamma} \times \ddot{\gamma}|}{|\dot{\gamma}|^3}$$

$\vec{B} = \vec{T} \times \vec{N} =$ binormal

$$\frac{d}{dt}\begin{pmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{pmatrix}$$

$$\tau = -B' \cdot N = +N' \cdot B = \frac{(\mathring{\gamma} \times \ddot{\gamma}) \cdot \dddot{\gamma}}{|\mathring{\gamma} \times \ddot{\gamma}|^2} = \frac{(\gamma' \times \gamma'') \cdot \gamma'''}{|\gamma' \times \gamma''|^2}$$

κ measure speed of rotation of $\vec{T}$

τ " " " " of $\vec{B}$

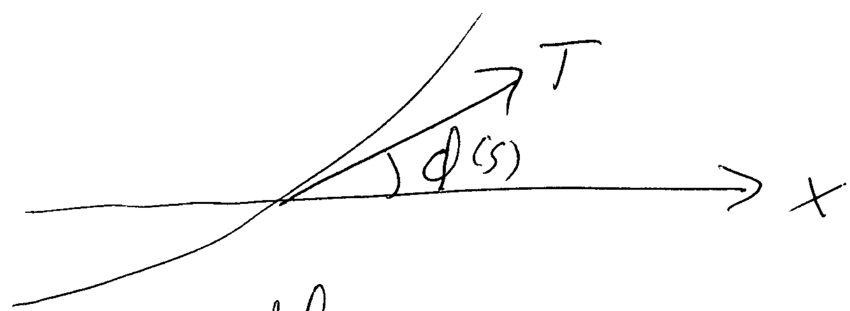
= " " " of tangent plane.

$(\kappa(s), \tau(s))$ determines curve up to direct isometry.

Plane curves.

$$N_s = (0,0,1) \times \vec{T} = 90^\circ \text{ ccw rotation of } \vec{T}$$

$$\frac{d}{ds}\begin{pmatrix}\vec{T} \\ \vec{N}_s\end{pmatrix} = \begin{pmatrix} 0 & K_s \\ -K_s & 0\end{pmatrix}\begin{pmatrix}\vec{T} \\ \vec{N}\end{pmatrix}$$



$$K_s = \frac{d\phi}{ds}$$

$K_s(s)$ determines curve up to direct isometry.

$\oint K_s \, ds$ = total signed curvature $= 2\pi n$

If <u>simple</u> closed curve, $\int K_s \, ds = \pm 2\pi$

$A \leq \frac{\ell^2}{4\pi}$ Isoperimetric (did not prove - out of time)

4 vertex thm: Every (convex) simple closed curve has at least 4 places where $\frac{dK_s}{ds} = 0$

Surfaces need <u>two</u> parameters (u,v) instead of one (t)

$\sigma: U \to \mathbb{R}^3$, $\{\sigma_u, \sigma_v\}$ linearly independent.

Tangent space $=$ Span $\{\sigma_u, \sigma_v\}$

Unit Normal vector $= \pm \dfrac{\sigma_u \times \sigma_v}{|\sigma_u \times \sigma_v|} = \pm N_\sigma$

If $(N_\sigma)_3 \neq 0$, then locally $z = f(x,y)$, f smooth.

Key tool: 1) IFT

If $f: \mathbb{R}^n \to \mathbb{R}^n$ is smooth, and $Df|_p$ is invertible, then f is a local diffeo near p and $D(f^{-1}) = (Df)^{-1}$

2) Local coordinates

$$\begin{tikzcd} S_1 \arrow[r, "f"] & S_2 \\ \cdot \arrow[u, "\sigma_1"] \arrow[r, "g"] & \cdot \arrow[u, "\sigma_2"'] \end{tikzcd}$$

Study g to understand f.