Space curve $\vec{\gamma}(s)$ unit speed.

$\vec{T} = \dot{\vec{\gamma}}$ — direction of motion

$\dot{\vec{T}} = \kappa \vec{N}, \quad N = \dfrac{\ddot{\vec{\gamma}}}{|\ddot{\gamma}|}$

$\vec{B} = \vec{T} \times \vec{N}$ defines plane of motion.

$\dot{\vec{B}} = -\tau \vec{N}$ $\qquad \tau =$ torsion.

$(t = s)$

$$\frac{d}{dt}\vec{T} = \kappa \vec{N}$$

$$\frac{d}{dt}\vec{N} = -\kappa \vec{T} + \tau \vec{B}$$

$$\frac{d}{dt}\vec{B} = -\tau \vec{N}$$

$$\frac{d}{dt}\begin{pmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{pmatrix} = \cancel{\begin{pmatrix} 0 & \kappa & 0 \\ \tau & 0 & -\tau \end{pmatrix}} \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{pmatrix}$$

Frenet – Serret equs.

Thm If two curves $\vec{\gamma}_1, \vec{\gamma}_2$ are related by a direct isometry, and if $\kappa_1(s) \neq 0$, then $\kappa_2(s) = \kappa_1(s)$ and $\tau_2(s) = \tau_1(s)$

Pf $\vec{\gamma}_2(s) = R\vec{\gamma}_1(s) + \vec{a}$

$$\vec{T}_2(s) = R\vec{T}_1$$

$$\dot{\vec{T}}_2 = R\dot{\vec{T}}_1. \qquad \kappa_2 = |\dot{\vec{T}}_2| = \kappa_1$$

$$\vec{N}_2 = \frac{\dot{\vec{T}}_2}{\kappa_2} = R\frac{\dot{\vec{T}}_1}{\kappa_1} = R\vec{N}_1$$

$$\vec{B}_2 = \vec{T}_2 \times \vec{N}_2 = (R\vec{T}_1) \times (R\vec{N}_1) = R(\vec{T}_1 \times \vec{N}_1) = R\vec{B}_1$$

$$\dot{\vec{B}}_2 = R\dot{\vec{B}}_1 = R(-\tau_1)\vec{N}_1 = -\tau_1 R\vec{N}_1 = -\tau_1 \vec{N}_2$$

So $\tau_2 = \tau_1$

Thm If two curves have the same curvature $(\neq 0)$ & torsion, they are related by a direct isometry

Pf WLOG, assume $\gamma_1(0) = \gamma_2(0) = 0$, $\dot{\gamma}_1(0) = \dot{\gamma}_2(0) = (1,0,0)$,

$\vec{N}_1(0) = \vec{N}_2(0) = (0,1,0)$,

$\vec{B}_1(0) = \vec{B}_2(0) = (0,0,1)$

$$\frac{d}{dt}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

Since $\kappa_2 = \kappa_1$, $\tau_2 = \tau_1$ same equs for $(\vec{T}_1, \vec{N}_1, \vec{B}_1)$ as for $(\vec{T}_2, \vec{N}_2, \vec{B}_2)$

Same initial conditions.

$\Rightarrow$ Same soln.

so $\vec{T}_2(s) = \vec{T}_1(s) \Rightarrow \dot{\gamma}_2(s) = \dot{\gamma}_1(s)$

$$\gamma_2(s) - \gamma_2(0) = \gamma_1(s) - \gamma_1(0)$$

$$\gamma_2(s) = \gamma_1(s)$$

<u>Thm</u> If $\kappa \neq 0$, then

$$\tau = \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \dddot{\gamma}}{|\dot{\gamma} \times \ddot{\gamma}|^2}$$

(formula applies to all regular curves, not just unit-speed)

<u>Pf</u> First suppose unit speed.

$$\dot{\vec{B}} = -\tau \vec{N}, \quad \text{so} \quad \tau = -\vec{N} \cdot \dot{\vec{B}}$$

$$\tau = -\vec{N} \cdot \frac{d}{ds}(\vec{T} \times \vec{N}) = -\vec{N} \cdot (\dot{\vec{T}} \times \vec{N} + \vec{T} \times \dot{\vec{N}})$$

$$= -\vec{N} \cdot (\vec{T} \times \dot{\vec{N}})$$

$$\vec{N} = \frac{1}{\kappa}\dot{\vec{T}} = \frac{1}{\kappa}\ddot{\gamma} \quad ; \quad \dot{\vec{N}} = \frac{1}{\kappa}\dddot{\gamma} - \frac{\dot{\kappa}}{\kappa^2}\ddot{\gamma}$$

$$= \frac{1}{\kappa}\dddot{\gamma} - \underbrace{\frac{\dot{\kappa}}{\kappa}\vec{N}}_{\text{does not contribute}}$$

$$\tau = -\vec{N} \cdot \left(\vec{T} \times \frac{1}{\kappa}\dddot{\gamma}\right) = \frac{-\ddot{\gamma} \cdot (\dot{\gamma} \times \dddot{\gamma})}{\kappa^2}$$

$$= \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \dddot{\gamma}}{|\dot{\gamma} \times \ddot{\gamma}|^2}$$

If γ isn't unit speed,

$$\tau = \frac{\gamma''' \cdot (\gamma' \times \gamma'')}{|\gamma' \times \gamma''|^2} \qquad \text{where } {}' = \frac{d}{ds}$$

$$\dot{} = \frac{d}{dt} = \left(\frac{ds}{dt}\right)\frac{d}{ds}$$

$$\dot\gamma = \frac{ds}{dt} \cdot \gamma'$$

$$\ddot\gamma = \left(\frac{ds}{dt}\right)^2 \gamma'' + \frac{d^2s}{dt^2}\gamma'$$

$$\dddot\gamma = \left(\frac{ds}{dt}\right)^3 \gamma''' + 3\left(\frac{ds}{dt}\right)\left(\frac{d^2s}{dt^2}\right)\gamma'' + \cancel{\left(\frac{ds}{dt}\right)^3}\frac{d^3s}{dt^3}\gamma'$$

$$\dot\gamma \times \ddot\gamma = \left(\frac{ds}{dt}\right)^3 (\gamma' \times \gamma'')$$

$$(\dot\gamma \times \ddot\gamma) \cdot \dddot\gamma = \left(\frac{ds}{dt}\right)^6 (\gamma' \times \gamma'') \cdot \gamma'''$$

$$\frac{\dddot\gamma \cdot (\dot\gamma \times \ddot\gamma)}{|\dot\gamma \times \ddot\gamma|^2} = \frac{\gamma''' \cdot (\gamma' \times \gamma'')}{|\gamma' \times \gamma''|^2} = \tau$$

Helix

$$\gamma(t) = (a\cos(t),\ a\sin(t),\ bt)$$

$$\dot{\gamma} = (-a\sin(t),\ a\cos(t),\ b)$$

$$\ddot{\gamma} = (-a\cos(t), -a\sin(t),\ 0)$$

$$\dddot{\gamma} = (a\sin t,\ -a\cos(t),\ 0)$$

$$\dot{\gamma} \times \ddot{\gamma} = (ab\sin(t),\ -ab\cos(t),\ a^2)$$

$$|\dot{\gamma} \times \ddot{\gamma}| = |a|\sqrt{a^2+b^2}$$

$$(\dot{\gamma} \times \ddot{\gamma}) \cdot \dddot{\gamma} = a^2 b$$

$$\tau = \frac{a^2 b}{a^2(a^2+b^2)} = \frac{b}{a^2+b^2}$$

$$\kappa = \frac{|a|}{a^2+b^2}$$

If $\tau = 0$, plane curve.

If $\tau = 0$ ~~and~~ and $\kappa =$ constant, circle

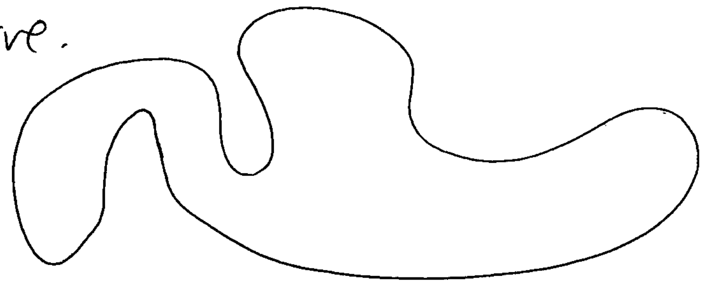
$$\frac{d}{ds}\left(\gamma + \frac{1}{\kappa}\vec{N}\right) = \dot{\gamma} + \frac{1}{\kappa}(-\kappa T + \tau B)$$
$$= \tau B = 0$$

$$\gamma + \frac{1}{\kappa}\vec{N} = \vec{a} = \text{constant}$$

~~a=~~ $|\gamma - \vec{a}| = \frac{1}{\kappa} = \text{constant}.$

On a sphere centered at $\vec{a}$.

Consider a simple closed plane curve.



Unit speed curve, periodic w/ period ℓ, no self-intersections.

Def: A vertex is a place where $\dot{\kappa}_s = 0$

Thm

1) $\int_0^\ell \kappa_s \, ds = \pm 2\pi$

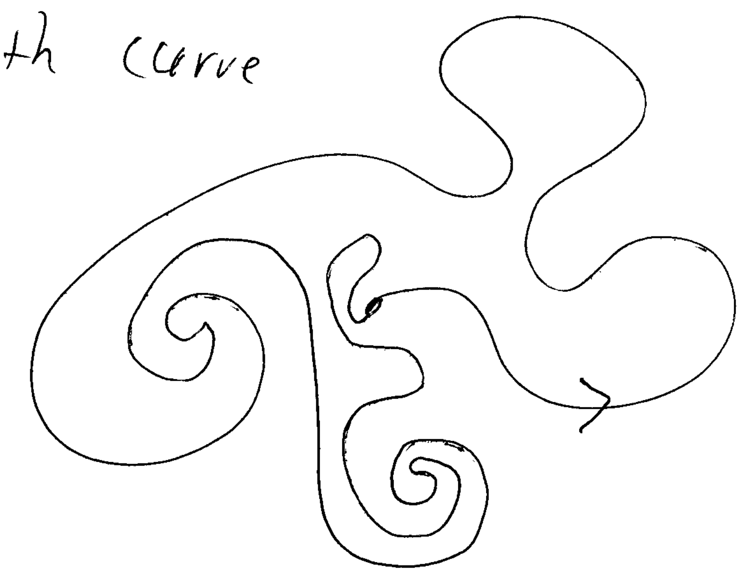
2) There are at least 4 vertices.
(4 vertex thm)

3) Area enclosed $= A(\gamma) \leq \frac{\ell^2}{4\pi}$ with equality only for circle.

Assume: Jordan curve thm:

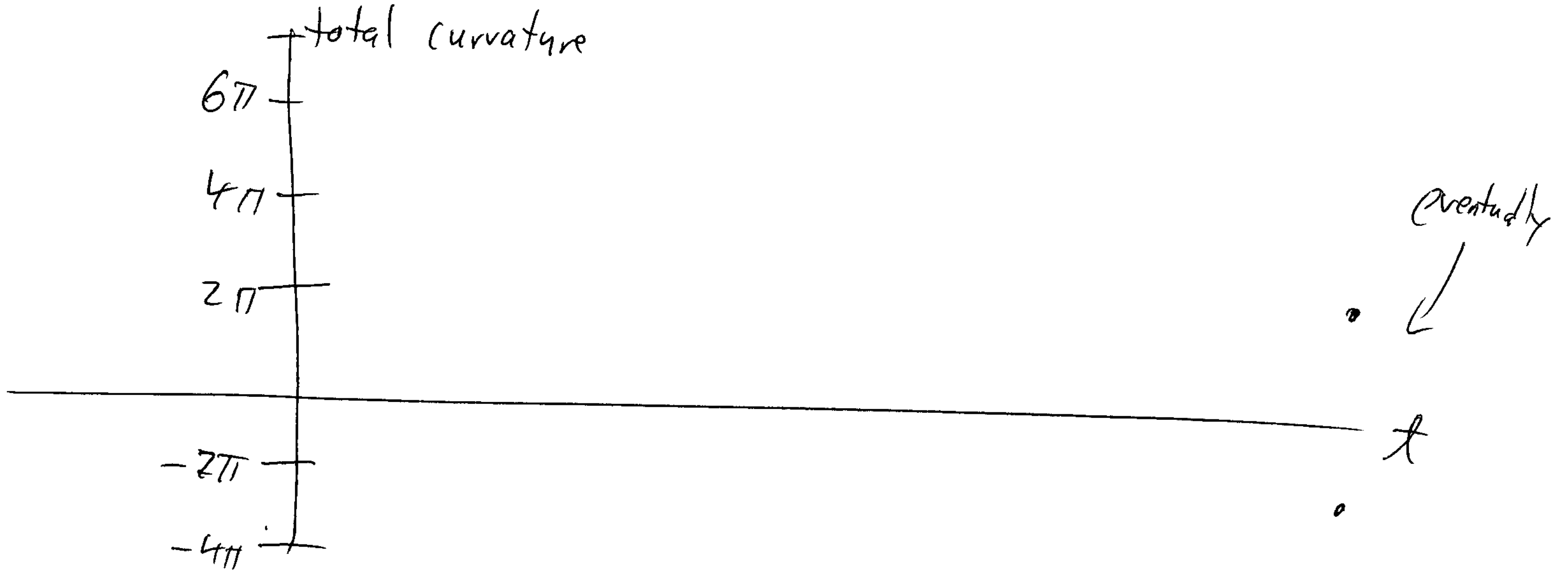
(A simple closed curve separates the plane into 2 pieces, the inside & the outside).

<u>Thm 1</u> Start with curve



Pump air into inside.

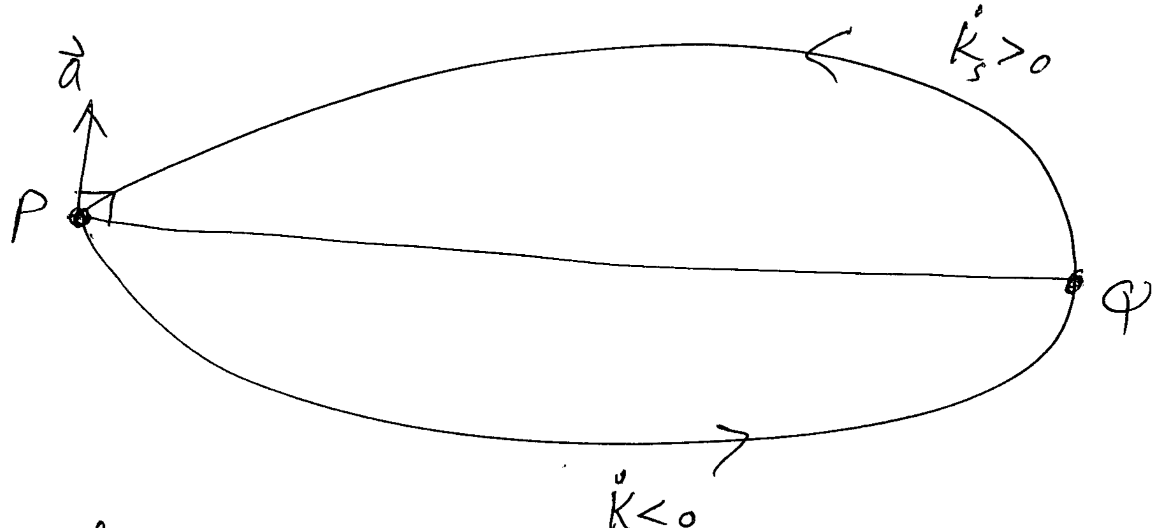
Curve gradually changes into circle.



Thm Every <u>convex</u> closed curve has at least 4 vertices.

Step 1: 2 vertices, P where $\kappa_s = \max$

Q where $\kappa_s = \min$.



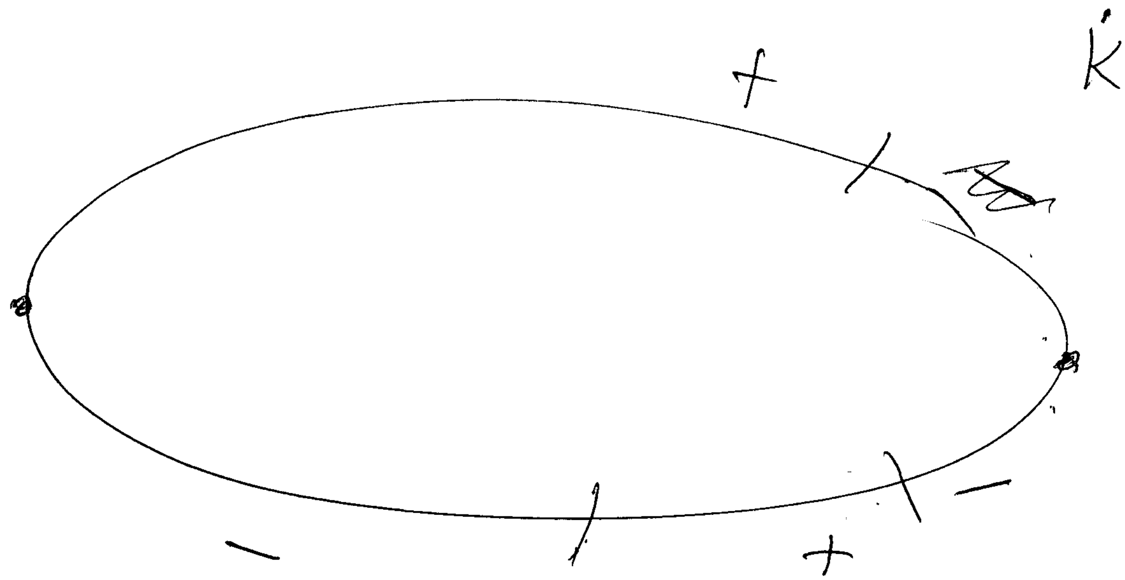
$$\int_0^{l} \dot{\kappa}_s \, (\gamma \cdot a) > 0$$

But, $\dot{N}_s = -\kappa_s \vec{T}$, so

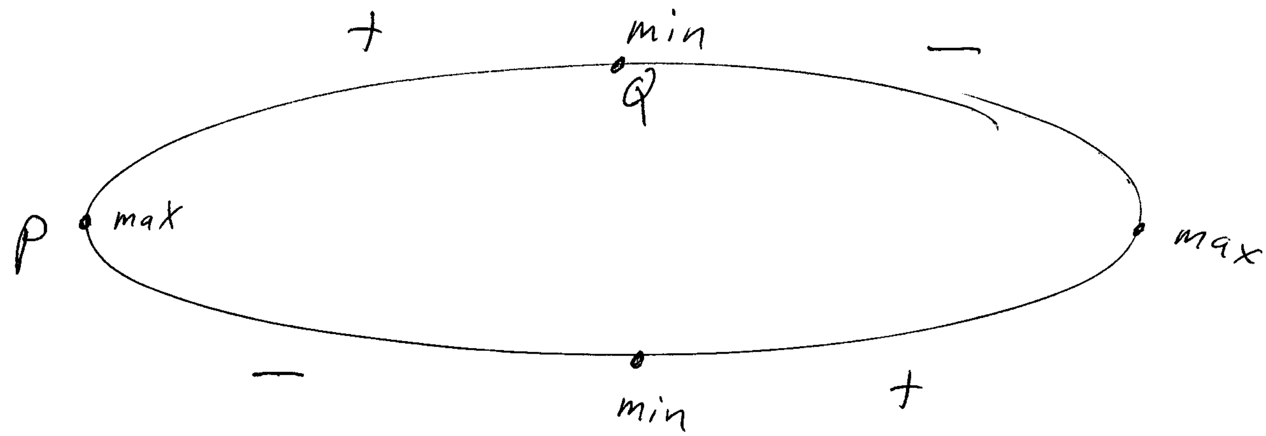
$$(\dot{\kappa}_s \gamma) = (\kappa_s \gamma)^{\cdot} - \kappa_s \dot{\gamma} = (\kappa_s \gamma + \vec{N}_s)^{\cdot}$$

$$\vec{a} \cdot (\dot{\kappa}_s \gamma) = \frac{d}{dt} \vec{a} \cdot (\kappa_s \gamma + \vec{N}_s)$$

$$\int \vec{a} \cdot (\dot{\kappa}_s \gamma) = \Delta \; \vec{a} \cdot (\kappa_s \gamma + \vec{N}_s) = 0$$



$\dot{K}$
+
−
−
+



+
min
−
Q
P
max
max
−
min
+