Level surface. $f: \mathbb{R}^3 \to \mathbb{R}$.

$$S = \{(x,y,z) \mid f(x,y,z) = 0\}$$

Ex: $x^2+y^2+z^2-1=0$ sphere.

$x+3y+5z-7=0$ plane
$(\nabla F = (1,3,5))$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$$ Ellipsoid.

$$\left(\nabla f = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right)\right)$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}-1=0$$ Hyperbaloid of 1 sheet

$$\left(\nabla f = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{-2z}{c^2}\right)\right)$$

$$\left.\begin{aligned} &\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}-1=0 \\ &\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}+1=0 \end{aligned}\right\}$$ Hyperbolid of 2sheets

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$ double cone
Surface away from origin.

~~Elliptic~~

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-z=0$$ Elliptic paraboloid

$$\nabla f=\left(\frac{2x}{a^2},\frac{2y}{b^2},-1\right)$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-z=0$$ Hyperbolic paraboloid.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$ Elliptic cylinder

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0$$ Hyperbolic cylinder

$$\frac{x^2}{a^2}-y=0$$ parabolic cylinder.

Thm If f is smooth and ∇f is never zero when $f=0$,

$\{f=0\}$ is a smooth surface.

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pf: Suppose $f(x_0, y_0, z_0) = 0$ and $\nabla f(x_0, y_0, z_0) = (a, b, c) \neq \vec{0}$

Suppose $c \neq 0$.

Define $h(x,y,z) = (x, y, f(x,y,z))$

$$Dh = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f_x & f_y & f_z \end{pmatrix} \qquad \det Dh = \frac{\partial f}{\partial z}.$$

At (x_0, y_0, z_0), $\det Dh = c \neq 0$.

By IFT, locally h has an inverse.

$$(x,y,z) = h^{-1}(x, y, f(x,y,z))$$

On the surface near (x_0, y_0, z_0),

$$(x,y,z) = h^{-1}(x,y,0)$$

$$z = \left(h^{-1}(x,y,0)\right)_3,$$ so $z =$ smooth function of x, y

A quadric surface is a level set of

$$x^T A x + b^T x + c \text{, where}$$

A is a symmetric 3×3 matrix, b is a 3-vector and c is a scalar.

$$c_1 x^2 + c_2 xy + c_3 xz + c_4 y^2 + c_5 yz + c_6 z^2 + c_7 x + c_8 y + c_9 z + c_{10} = 0$$

<u>Thm</u> Every quadric "surface" is a direct isometry of:

plane	line,
ellipsoid	pt.
parabaloid	2 intersecting planes.
hyperboloid	
cone,	
cylinder,	
2 planes	

Pf By doing a rotation, make the e-vecs of A be the coordinate axes, so

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \qquad \lambda_1 \geq \lambda_2 \geq \lambda_3$$

$$f(x) = (\lambda_1 x^2 + b_1 x) + (\lambda_2 y^2 + b_2 y) + (\lambda_3 z^2 + b_3 z) + c$$

If $\lambda_1 > 0$ or $\lambda_1 < 0$,

$$\lambda_1 x^2 + b_1 x = \lambda_1 \left(x^2 + \frac{b_1}{\lambda_1} x\right)$$

$$= \lambda_1 \left[\left(x + \frac{b_1}{2\lambda_1}\right)^2 - \frac{b_1^2}{4\lambda_1^2}\right]$$

$$= \lambda_1 \left(x + \frac{b_1}{2\lambda_1}\right)^2 - \frac{b_1^2}{4\lambda_1}$$

After isometry,

$$f(x,y,z) = (\lambda_1 x^2 \text{ or } b_1 x \text{ or } 0) + (\lambda_2 y^2 \text{ or } b_2 y \text{ or } 0) + (\lambda_3 z^2 \text{ or } b_3 z \text{ or } 0) + c.$$