Inverse function thm in 1D.

$f: \mathbb{R} \to \mathbb{R}$ (smooth)

If $f'(x_0) \neq 0$, then there is a nbhd V of x_0 ~~wher~~ s.t. $f\big|_V : V \to f(V)$ is a diffeo, and

$$\frac{d f^{-1}}{dy}(f(x_0)) = \frac{1}{f'(x_0)}$$

Let $f'(x_0) = a$

$$y - y_0 \approx a(x - x_0)$$

$$x - x_0 \approx \frac{1}{a}(y - y_0)$$

$$x = f^{-1}(y) \approx x_0 + \frac{1}{a}(y - y_0)$$

Real proof: Since f' is continuous, pick nbhd V where $.9a < f' < (1.1)a$.

$c \in (x_1, x_2)$

MVT: $f(x_1) - f(x_2) = (x_1 - x_2) f'(c) = (x_1 - x_2)a + \text{error}$

with $|\text{error}| < \frac{1}{10} |(x_1 - x_2)a|$

Solve $y = f(x)$

$$x_1 = x_0 + \frac{(y - y_0)}{a}$$

$$x_{i+1} = x_i + \frac{(y - f(x_i))}{a}$$

$$|y - f(x_{i+1})| < \frac{1}{10} |y - f(x_i)|$$

$$y - f(x_{i+1}) = y - \left(f(x_i) + a(x_{i+1} - x_i) + \text{error}\right)$$

$$= y - (f(x_i) + y - f(x_i) + \text{error})$$

$$= \quad |\text{error}| < \left|\frac{1}{10} a(x_{i+1} - x_i)\right| = \left|\frac{1}{10} (y - f(x_i))\right|$$

$$x = \lim x_i$$

$$f(x) = \lim f(x_i) = y$$

IFT in n-dimensions

$f: \mathbb{R}^n \to \mathbb{R}^n$.

$$Df = \left(\frac{\partial f^i}{\partial x_j} \right)$$ is best linear approx.

$$\vec{f}(\vec{x}) - \vec{f}(\vec{x}_0) = \left(Df\big|_{x_0} \right) (\vec{x} - \vec{x}_0) + \text{error}.$$

Thm If $Df\big|_{x_0}$ is invertible, then there is a nbhd V of x_0 s.t. $f: V \to f(V)$ is a diffeo, and $Df^{-1}\big|_{f(x_0)} = \left(Df\big|_{x_0} \right)^{-1}$

"Pf" Pick V where $Df \approx Df\big|_{x_0}$

Then $f(x_1) - f(x_2) = \left(Df\big|_{x_0} \right) (\vec{x}_1 - \vec{x}_2) + \text{error}.$

with $|\text{error}| < \frac{1}{10} \left| Df\big|_{x_0} (\vec{x}_1 - \vec{x}_2) \right|$

Solve $\vec{y} = \vec{f}(\vec{x})$

$$A = Df\big|_{x_0}$$

$$\vec{x}_1 = \vec{x}_0 + \cancel{Df\big|_{x_0}}\, A^{-1}(\vec{y} - \vec{y}_0)$$

$$\vec{x}_{i+1} = \vec{x}_i + A^{-1}(y - f(\vec{x}_i))$$

$$|\vec{y} - \vec{f}(\vec{x}_{i+1})| < \frac{1}{10} |\vec{y} - \vec{f}(\vec{x}_i)|$$

$$\vec{x} = \text{Lim } \vec{x}_i$$

$$\vec{f}(\vec{x}) = \vec{y}.$$

$f: \mathbb{R}^n \to \mathbb{R}^m$

$\cancel{n \geq m}$.

$n < m$.

$$A = Df\big|_{x_0} = \begin{array}{cc} & n \\ m & \begin{pmatrix} & \\ & \end{pmatrix} \end{array}$$

Assume.
rank $= n$

~~Then~~

~~Imm~~ <u>Immersion Thm.</u>

Suppose $Df\big|_{x_0} = A$ has rank n. Then
$\exists$ nbhd V of x_0 s.t., on $f(V)$,
$(m-n)$ variables are smooth functions of n variables.

<u>Pf</u> There are n lin-ind rows of A.
Suppose they are first n rows.

$\cancel{F: \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^{n-m}} \quad \cancel{\mathbb{R}^m \to \mathbb{R}^m}$

$\cancel{\mathbb{R}^n \times \mathbb{R}^{m-n}}$

$$\cancel{F(x,y) = f(x) + \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} \vec{u} \\ \vec{v} \end{pmatrix}}$$

Projection on 1st n entries.

~~fo:~~
$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "f"] \arrow[rr, bend right, "g"'] & \mathbb{R}^m \arrow[r, "\pi"] & \mathbb{R}^n \end{tikzcd}$$

$Dg =$ first n rows of Df

$=$ invertible

So g has an inverse.

1st n variables

~~X~~ $f(x) = \begin{pmatrix} y \\ z \end{pmatrix}$

last $m-n$

$y = g(x)$

$x = g^{-1}(y)$

$z =$ last $m-n$ terms $\circ\, f \circ g^{-1}(y)$

Implicit Function Thm.

Suppose $m < n$, $f: \mathbb{R}^n \to \mathbb{R}^m$,

$Df\big|_{X_0}$ has rank m.

Specifically, suppose first m cols of Df are lin ind.

~~Then~~ Suppose $f(X_0) = 0$.

Then $\{\vec{f}(\vec{x}) = 0\} \cap \left(\text{nbhd of } X_0\right)$

is an $(n-m)$-dim'l surface, with

first m variables $=$ funct (last $(m-n)$)

$\vec{x} = \binom{X}{Y}$

$F: \mathbb{R}^n \to \mathbb{R}^n$

$F\binom{X}{Y} = \binom{f(X,Y)}{Y}$

$$DF = \begin{array}{cc} \left(\begin{array}{c|c} \multicolumn{2}{c}{Df} \\ \hline 0 & I \end{array}\right) & \begin{array}{l} m \\ n-m \end{array} \\ \begin{array}{cc} m & n-m \end{array} & \end{array}$$

$\binom{X}{Y} = F^{-1}\binom{f(X,Y)}{Y}$

On $\{f(\vec{x}) = 0\}$, $\binom{X}{Y} = F^{-1}\binom{0}{Y}$; $X =$ 1st m coordinates of $F^{-1}\binom{0}{Y}$

Inverse Function Thm

If $f: \mathbb{R}^n \to \mathbb{R}^n$ and Df has maximal rank, f locally has an inverse

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Immersion thm. ~~If~~ $(m = n+k)$

If $f: \mathbb{R}^n \to \mathbb{R}^{n+k}$ and Df has maximal rank, then image of f is a graph: n free variables.
k variables that depend on them.

Implicit function thm: $(n = m+k)$

If $f: \mathbb{R}^{m+k} \to \mathbb{R}^m$, $f(\vec{x}_0) = 0$, and $Df|_{\vec{x}_0}$ has maximal rank, then locally $\{x \mid f(\vec{x}) = 0\}$ is graph of smooth $g: \mathbb{R}^k \to \mathbb{R}^m$.

If $\gamma: \mathbb{R} \to \mathbb{R}^3$ and $\dot{\gamma}(t_0) \neq 0$, then

locally $(y, z) = f(x)$

or $(x, z) = g(y)$

or $(x, y) = h(z)$

Immersion thm, $n=1$, $m=3$

<u>Plane level sets.</u> If $\vec{\nabla} F \neq 0$ when $F=0$, then

$F(x,y)=0$ is a smooth curve

If $\sigma_u \times \sigma_v \neq 0$, one of (x,y,z) = function of other two.

$\sigma: \mathbb{R}^2 \to \mathbb{R}^3$

$\left(\vec{\sigma}_u \times \vec{\sigma}_v \neq 0\right) \Longleftrightarrow D\sigma$ has rank 2.

$$D\sigma = \begin{pmatrix} \sigma_u & \sigma_v \end{pmatrix}$$

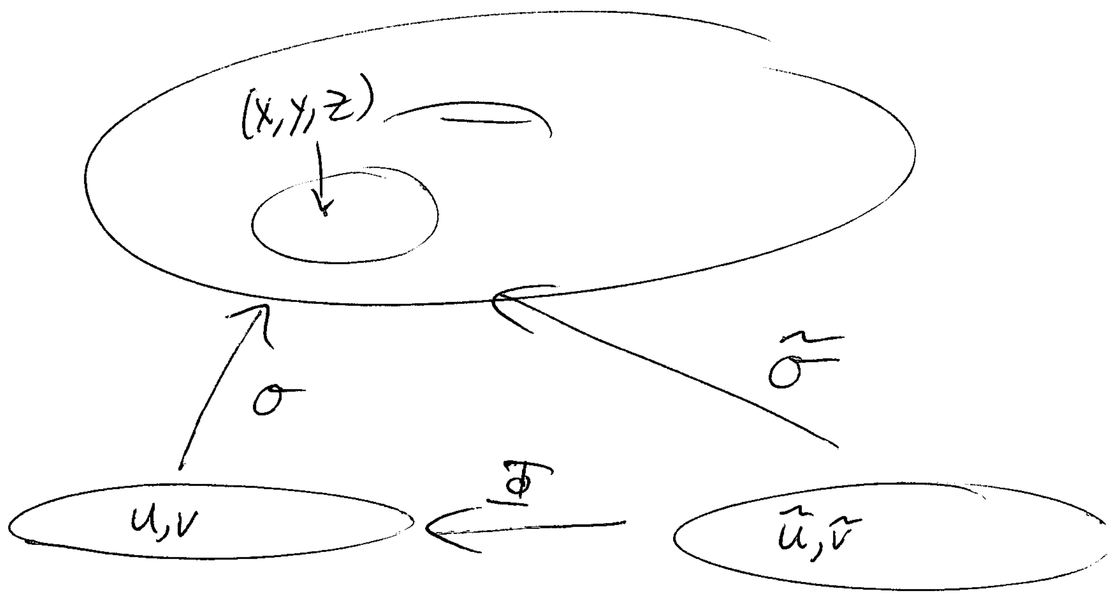
Immersion Thm

$\mathbb{R}^2 \to \mathbb{R}^3$

Level sets in $\mathbb{R}^3$. If $F: \mathbb{R}^3 \to \mathbb{R}$ and

$\nabla F \neq 0$ when $F=0$, then $\{\vec{x} \mid F(\vec{x})=0\}$ is a smooth surface.

$$DF = (\ \vec{\nabla} F\ )$$



Φ is local diffeo, as long as $D\sigma$ and $D\tilde{\sigma}$ have rank 2.

$D\sigma$ has rank 2, so $(u,v) \to (x,y)$ is local diffeo

$(u,v) =$ smooth function of (x,y)

$=$ Smooth function of part of $\tilde{\sigma}(\tilde{u},\tilde{v})$

$=$ Smooth " of $\tilde{u},\tilde{v}$