3 pictures of curves:

1) Graph of smooth function(s)

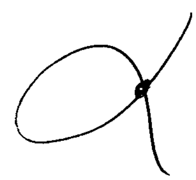
2) Level set

3) $\gamma : \mathbb{R} \to \mathbb{R}^{2 \text{ or } 3}$

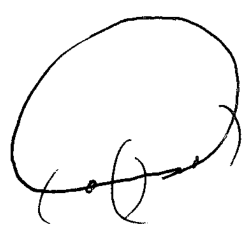
Want $\dot{\gamma} \neq 0$.

Locally, curve looks like piece of $\mathbb{R}$.

Slight problems: a)



b)

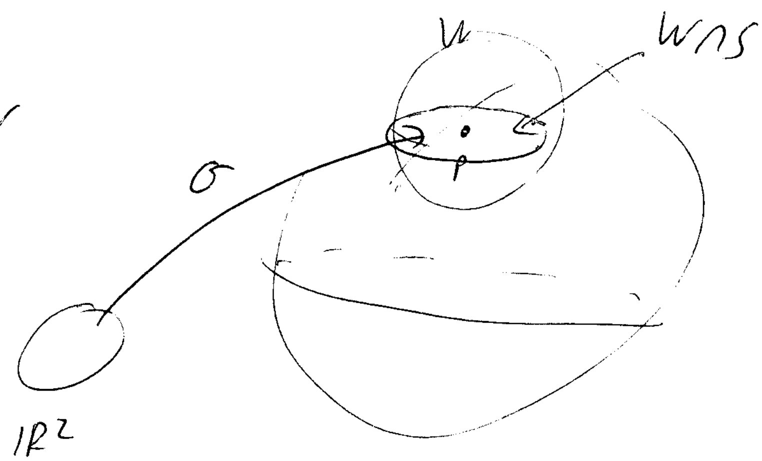


For surfaces, <u>must</u> work locally.

A surface is a subset of $\mathbb{R}^3$ that locally looks like piece of $\mathbb{R}^2$.

<u>Def</u> Let $S \subset \mathbb{R}^3$. S is a <u>surface</u> if for each $p \in S$, $\exists$ open sets $U \subset \mathbb{R}^2$, $W \subset \mathbb{R}^3$, $p \in W$, s.t. there are continuous maps

$$U \underset{\sigma^{-1}}{\overset{\sigma}{\rightleftarrows}} S \cap W$$

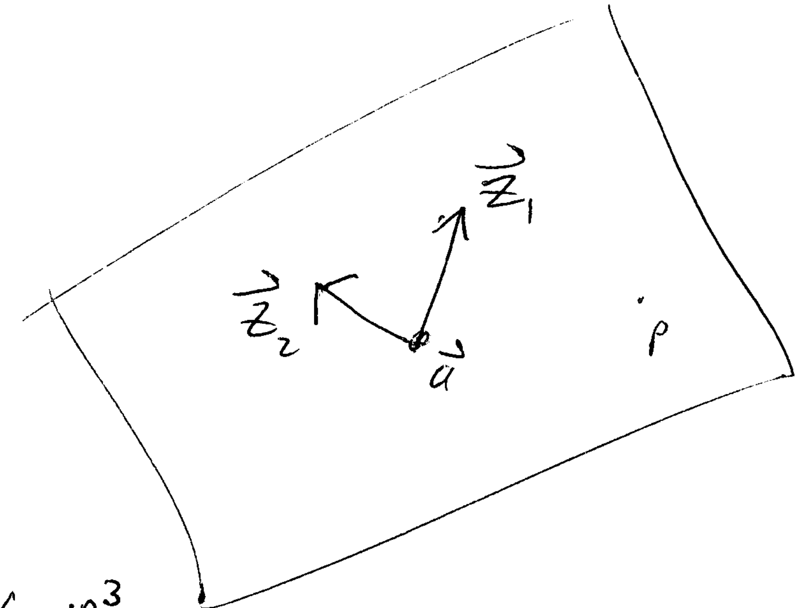


σ is called a <u>surface patch</u>

{patches} = atlas.

Note: A cube is a surface!

Ex1: Plane



$U = \mathbb{R}^2$ $\quad$ $W = \mathbb{R}^3$

(u,v) $\quad$ (x,y,z)

$$\sigma(u,v) = \vec{a} + u\,\vec{z}_1 + v\,\vec{z}_2 = (\vec{x})$$

$\begin{pmatrix} u \\ v \end{pmatrix}$ = coordinates of $\vec{x} - \vec{a}$ in basis $\{\vec{z}_1, \vec{z}_2\}$.

Ex2: Plane $ax + by + cz = d$. $\quad U = \mathbb{R}^2,\ W = \mathbb{R}^3$

If $c \neq 0$, $\quad z = \dfrac{d - ax - by}{c} \quad \sigma_1(u,v) = \left(u, v, \dfrac{d - au - bv}{c}\right)$

If $b \neq 0$ $\quad y = \dfrac{d - ax - cz}{b} \quad \sigma_2(u,v) = \left(u, \dfrac{d - au - cv}{b}, v\right)$

If $a \neq 0$ $\quad x = \dfrac{d - by - cz}{a} \quad \sigma_3(u,v) = (\sim, u, v)$

Ex3 $S = S^2 = \{x^2+y^2+z^2=1\}$.

$U_1 = (-\frac{\pi}{2}, \frac{\pi}{2}) \times (0, 2\pi)$

latitude longitude

$\sigma(u,v) = (\cos u \cos v, \cos u \sin v, \sin u)$

Missing poles & prime meridian.

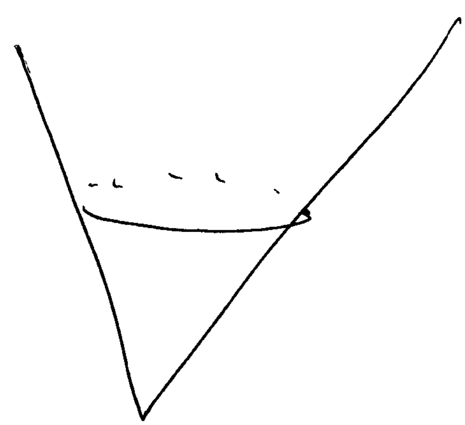
$U_2 = (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\pi, \pi)$

miss poles & int'l date line.

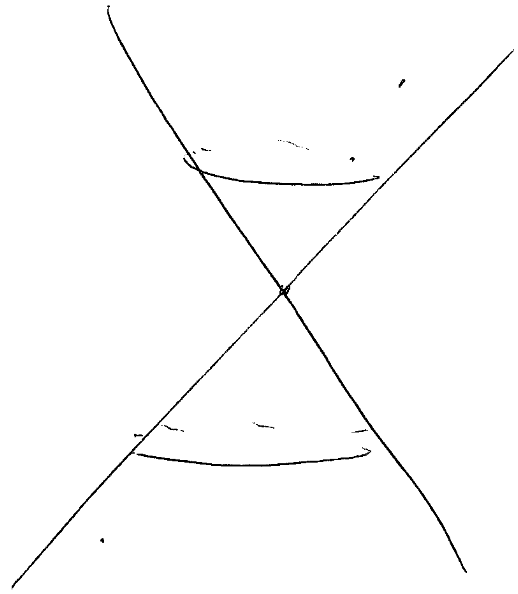
To get poles, swap $x \to z$.

Ex4 $\frac{1}{2}$ Cone $x^2+y^2=z^2$, $z \geq 0$

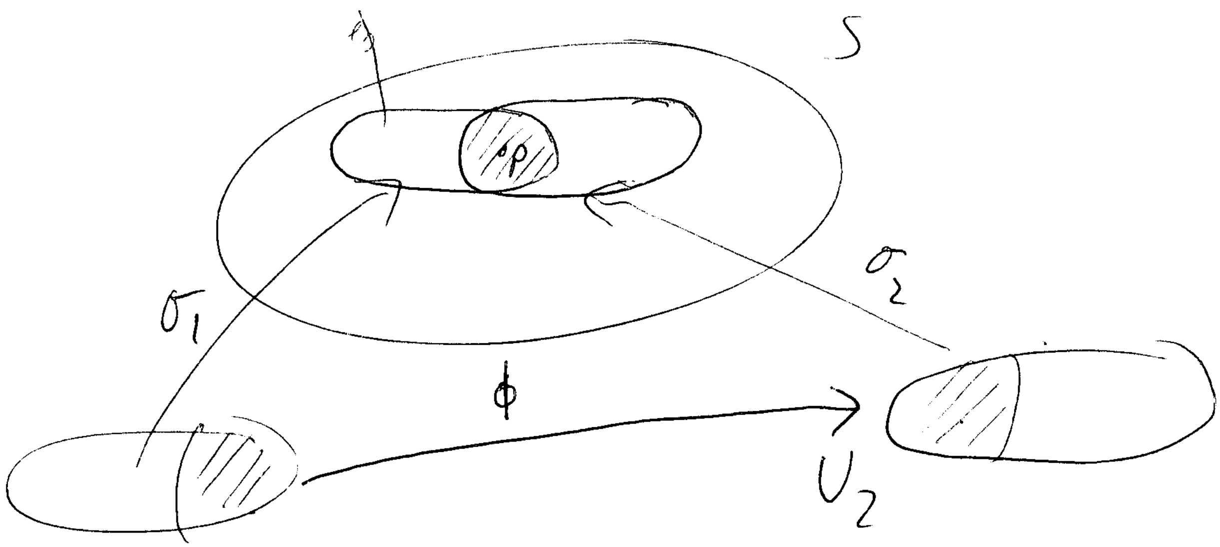
$\sigma(u,v) = (u, v, \sqrt{u^2+v^2})$



Cone $x^2+y^2+z^2$



Not a surface.
(Problems at origin)



<u>transition map</u> ϕ or <u>change-of-coordinates</u>

$= \sigma_2^{-1} \circ \sigma_1$ is continuous

$\phi^{-1} = \sigma_1^{-1} \circ \sigma_2$ is continuous.

What is a <u>smooth</u> surface?

1) Locally, graph of a smooth function

$z = f(x,y)$ or $y = g(x,z)$ or $x = h(y,z)$

2) Surface with smooth charts

with $\frac{\partial \sigma}{\partial u}$ and $\frac{\partial \sigma}{\partial v}$ linearly independent

(equivalently, $\frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v} \neq 0$)

3) Level set $F(x,y,z) = c$ of a smooth function F s.t.

$\nabla F \neq 0$ on S.

Thm $1 \Leftrightarrow 2$.

$\Rightarrow$ let $\sigma(u,v) = (u, v, f(u,v))$

$\sigma_u := \frac{\partial \sigma}{\partial u} = (1, 0, f_u)$

$\sigma_v = (0, 1, f_v)$ $\qquad \sigma_u \times \sigma_v = (-f_u, -f_v, 1)$

$\Leftarrow$ Suppose σ = smooth chart with

$\sigma_u \times \sigma_v \neq 0$. Suppose $(\sigma_u \times \sigma_v)_3 \neq 0$.

But $(\sigma_u \times \sigma_v)_3 = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \neq 0$

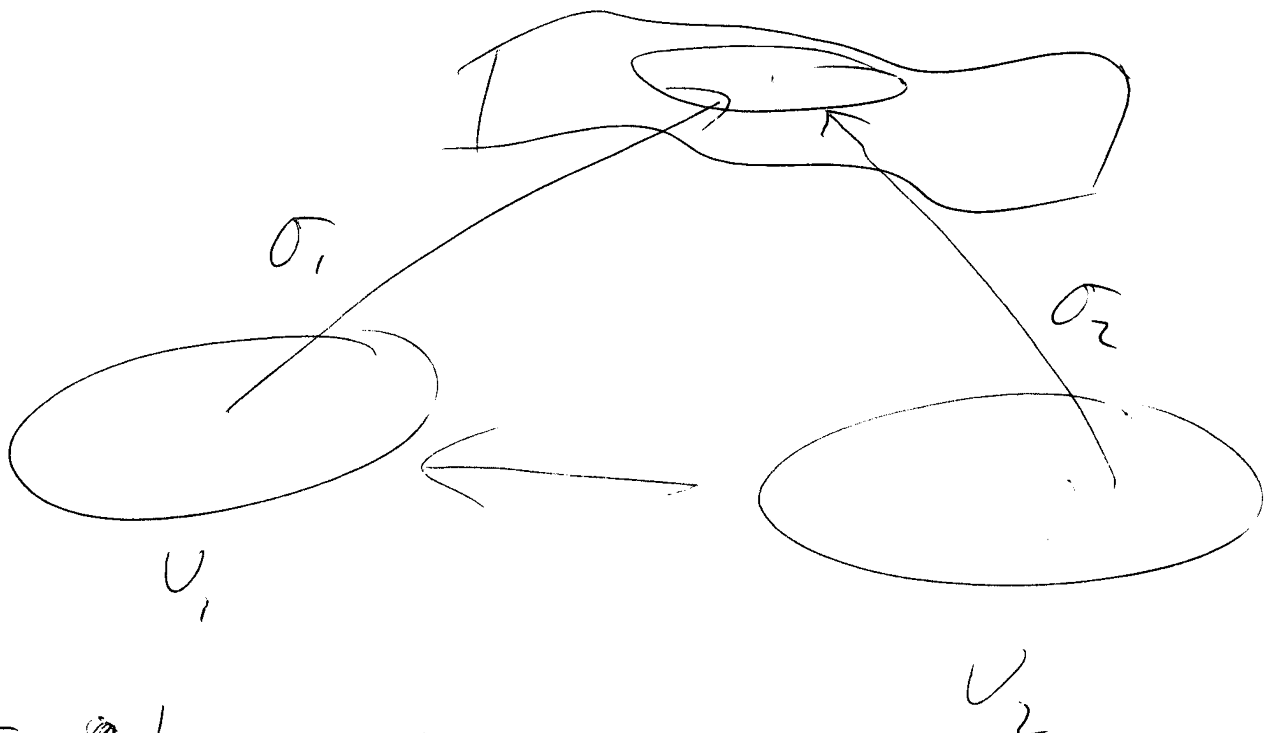
By inverse function thm, $(u,v) = h(x,y)$ for some smooth h.

$(x,y,z) = \sigma(h(x,y))$

~~let~~ So $z = \sigma^{(3)}(h(x,y)) := f(x,y)$

Thm If ~~σ is~~ S is a smooth surface (σ's are smooth w/ $\sigma_u \times \sigma_v \neq 0$), then the transition functions are smooth.

pf



σ_1 has property that

$(\sigma_1)_u \times (\sigma_1)_v \neq 0$. Suppose 3rd component is nonzero.

Then $(u_1, v_1) =$ smooth function of x, y

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = h\begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{But} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \sigma_2\begin{pmatrix} u_2 \\ v_2 \end{pmatrix}, \text{ so}$$

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \cancel{\text{first 2 comp}}\ h\left(\text{1st 2 components of } \sigma_2\begin{pmatrix} u_2 \\ v_2 \end{pmatrix}\right)$$

Thm $1 \Leftrightarrow 3$.

$$\Rightarrow \quad F(x,y,z) = z - f(x,y)$$
$$\nabla F = (-f_x, -f_y, 1) \neq 0.$$

$\Leftarrow$ Suppose $\nabla F = (a,b,c) \neq \vec{0}$.

If $c \neq 0$, let

$$H\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ F(x,y,z) \end{pmatrix}$$

$$DH = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ F_x & F_y & F_z \end{pmatrix}$$

$\det DH = F_z = c \neq 0$. So H^{-1} exists and is smooth.

On surface, $F=0$, so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = H^{-1}\begin{pmatrix} x \\ y \\ 0 \end{pmatrix},$$ so $z =$ 3-rd entry in $H^{-1}(x,y,0)$

Cor $2 \Leftrightarrow 3$.