Inverse function theorem in $\mathbb{R}^n$.

Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map

Suppose $f(p) = q$. If $Df_p = \begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f^n}{\partial x_1} & \cdots & \frac{\partial f^n}{\partial x_n} \end{pmatrix}$

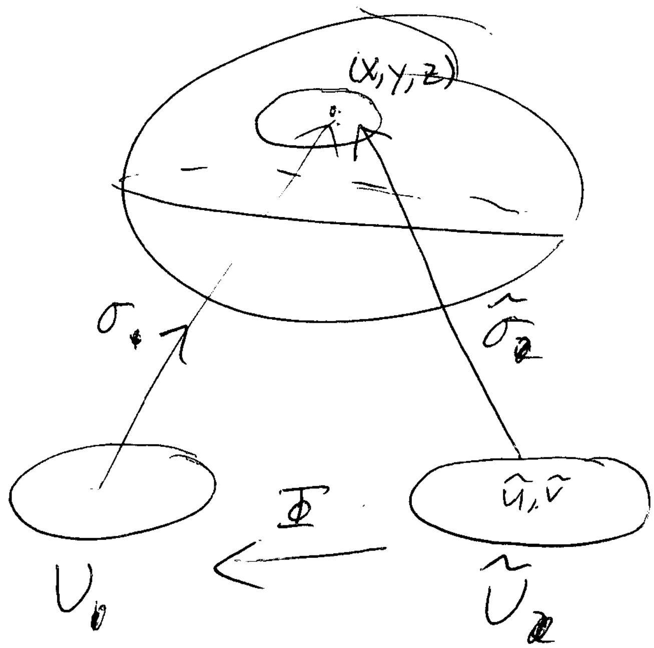
is invertible, then there is a nbhd U of p, and a nbhd V of q $= f(U)$ s.t. $f|_U : U \to V$

has a smooth inverse $g = f^{-1}$, $: V \to U$

Furthermore, $(Dg)_q = ((Df)_p)^{-1}$

$y = f(x)$,
$x = g(y)$

$(Df)_p = \left(\frac{dy}{dx}\right)_p$; $(Dg)_q = \frac{dx}{dy} = \frac{1}{dy/dx}$
$= \left(\frac{dy}{dx}\right)^{-1}$



$$\begin{pmatrix} u \\ v \end{pmatrix} = \Phi \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}$$

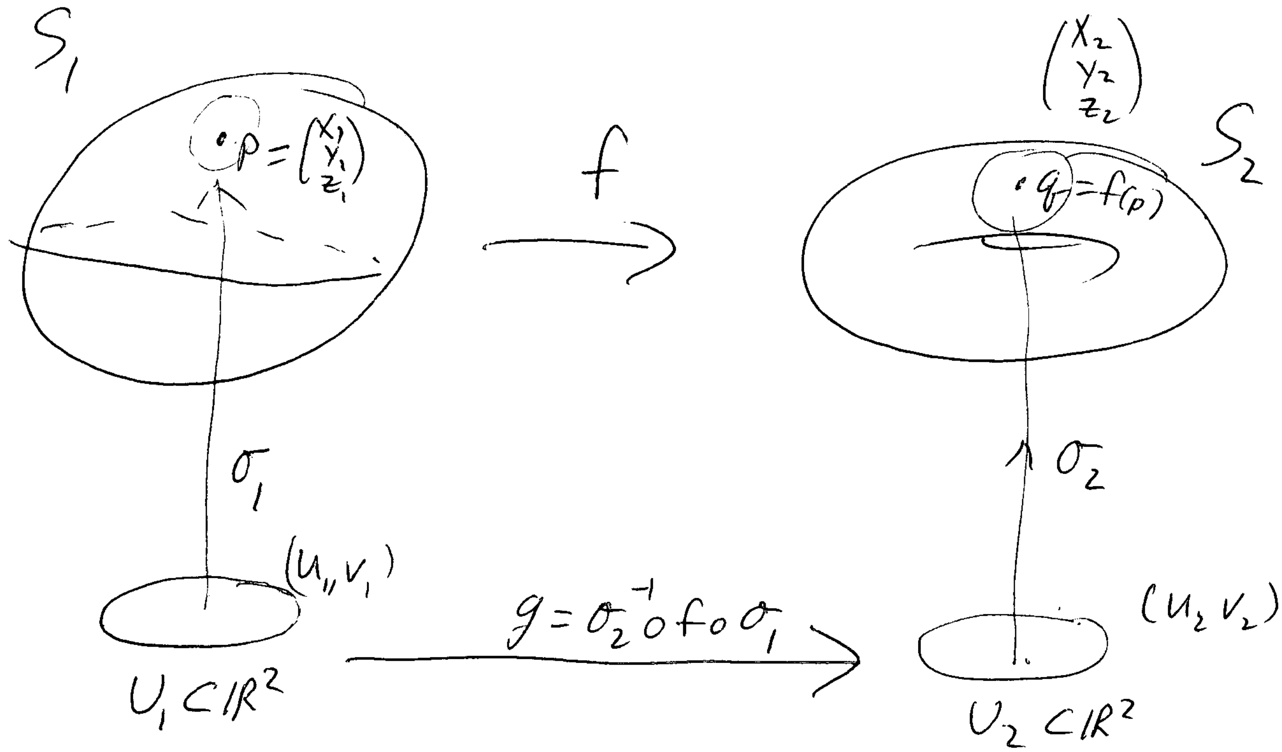
$$(u, v) = \Phi(\tilde{u}, \tilde{v})$$

$$\Phi = \sigma^{-1} \circ \tilde{\sigma} \quad : \mathbb{R}^2 \to \mathbb{R}^2$$

<u>Thm</u> If $\sigma, \tilde{\sigma}$ are regular patches, $\sigma_u \times \sigma_v \neq 0$
$\tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}} \neq 0$

then Φ is smooth.

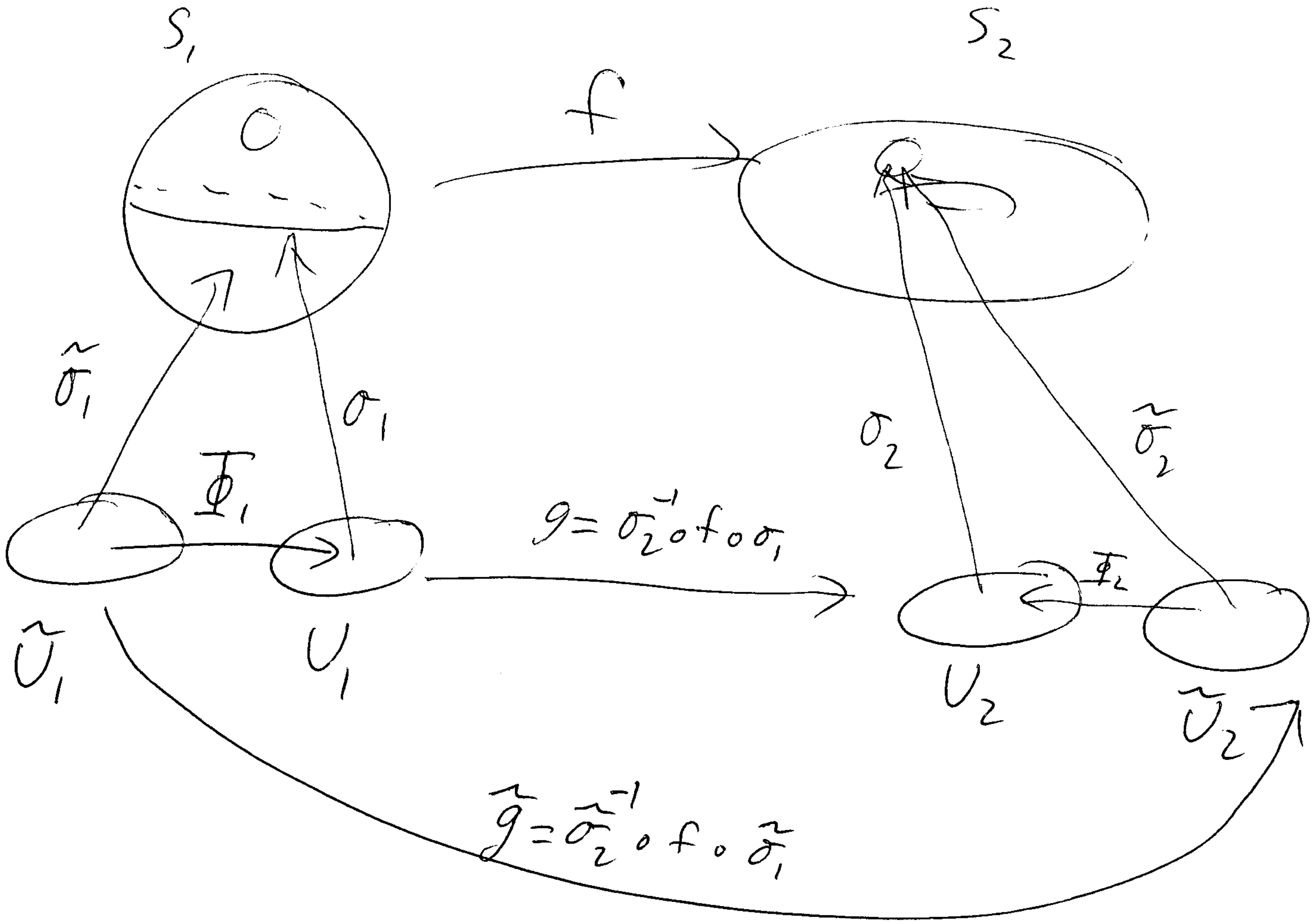
<u>Lemma:</u> If $\sigma_u \times \sigma_v$ has 3rd component nonzero, then $(u, v) =$ smooth function of (x, y)



To study $f: S_1 \to S_2$, study $\sigma_2^{-1} \circ f \circ \sigma_1 = g$

$$(u_2, v_2) = g(u_1, v_1)$$

We say f is smooth iff g is smooth.



$$= (\tilde\sigma_2^{-1} \circ \sigma_2) \circ \sigma_2^{-1} \circ f \circ \sigma_1 \circ \sigma_1^{-1} \circ \tilde\sigma_1$$

$$= \Phi_2^{-1} \circ g \circ \Phi_1$$

A <u>diffeomorphism</u> is a smooth map with a smooth inverse.

A local diffeomorphism is a smobth map ~~m~~ ~~that~~ s.t., for each $p \in S_1$ $\exists U \subset S_1$
$S_1 \to S_2$

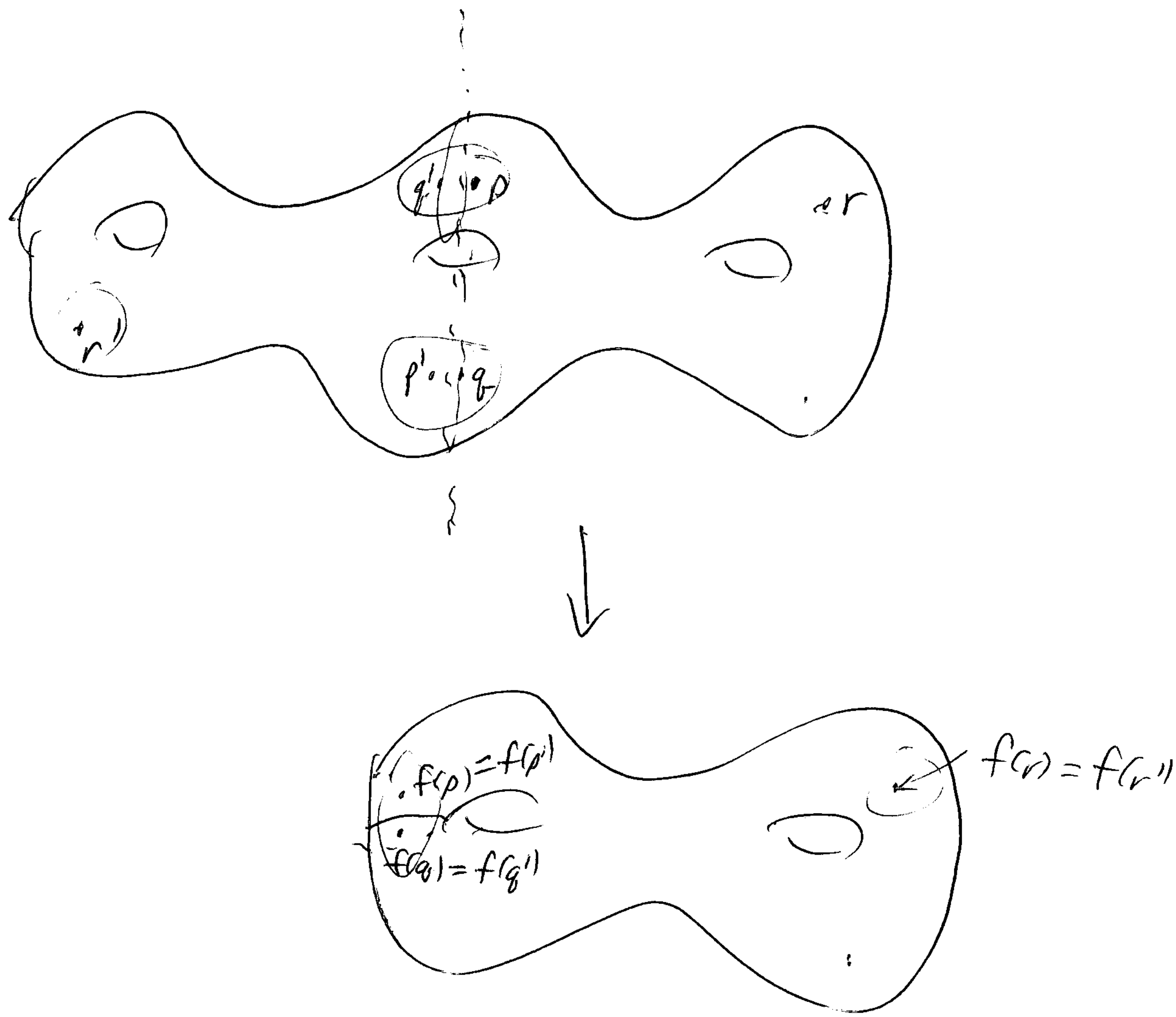
s.t. $f|_U$ is a diffeo $U \to f(U)$

Ex: $S_1 = \mathbb{R}^2 \subset \mathbb{R}^3$ $\quad S_2 =$ cylinder $(x^2+y^2=1)$

$$f(\overset{\cancel{u}\;v}{x,y},0) = (\cos(x), \sin(x), y)$$

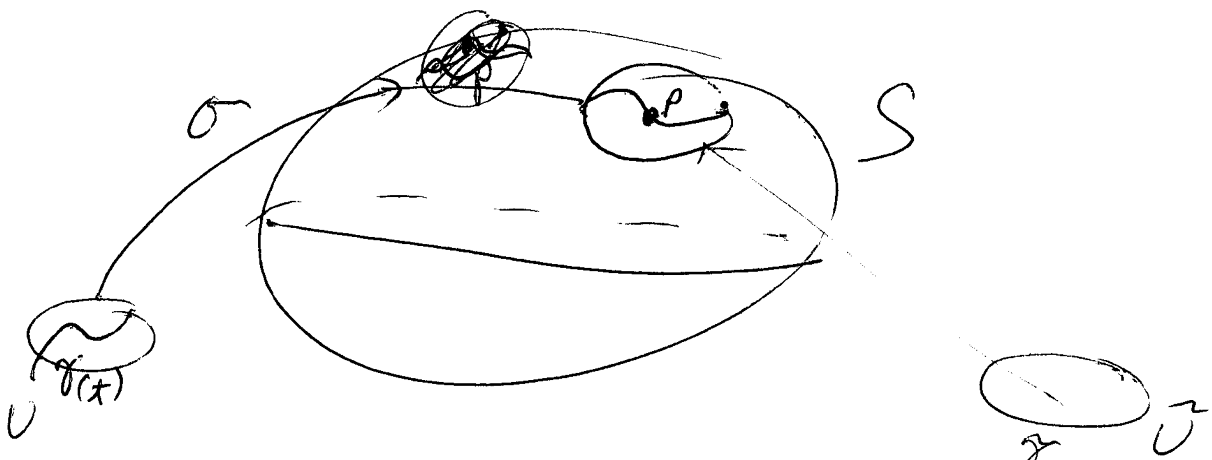
Not diffeo, not 1-1.

But <u>local</u> diffeo(morphism)



q'
p
q
r
r'
p'
q
f(p) = f(p')
f(q) = f(q')
f(r) = f(r')

Tangent~~s~~ spaces



$T_p S$ = { all possible velocities of curves on S, through p }

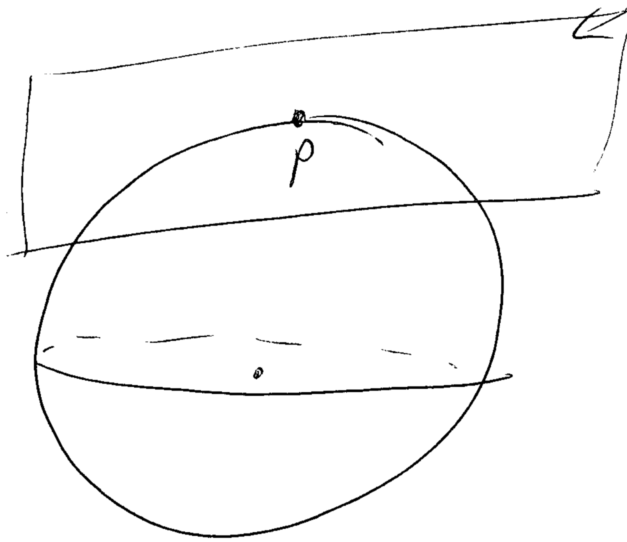
= tangent space to S at p.

$$\frac{d}{dt}(\sigma\circ\gamma) = \frac{\partial\sigma}{\partial u}\frac{du}{dt} + \frac{\partial\sigma}{\partial v}\frac{dv}{dt}$$

$$= \sigma_u \dot{u} + \sigma_v \dot{v}$$

$T_p S = \mathrm{Span}\{\sigma_u, \sigma_v\}$ = property of S, not of σ.

$d \quad \tilde{\sigma}\circ\tilde{\gamma} = \sigma\circ(\sigma^{-1}\circ\tilde{\sigma})\circ\tilde{\gamma} = \sigma\circ(\Phi\circ\tilde{\gamma})$



bad picture of $T_p S$

$(p, \vec{v})$

Tangent <u>bundle</u> of S = all pairs $(p, \vec{v})$ with $\vec{v}$ living at p.

A tangent vector has
1) a location.
2) a velocity.

Can only add tangent vectors w/ same location.

How to describe tangent plane at p.

1) Span $\{\sigma_u, \sigma_v\}$ Very coordinate-dependent.
2) By normal vector $\sigma_u \times \sigma_v$. defined up to scale.
3) By <u>unit</u> normal vector $\frac{\sigma_u \times \sigma_v}{|\sigma_u \times \sigma_v|} = \vec{N}_\sigma$ Sign?

$$(u,v) = \Phi(\tilde u, \tilde v)$$

$$\frac{\partial}{\partial u} = \frac{\partial \tilde u}{\partial u}\frac{\partial}{\partial \tilde u} + \frac{\partial \tilde v}{\partial u}\frac{\partial}{\partial \tilde v}$$

$$\begin{pmatrix}\vec\sigma_u\\ \vec\sigma_v\end{pmatrix} = \begin{pmatrix} \frac{\partial\tilde u}{\partial u} & \frac{\partial \tilde v}{\partial u}\\ \frac{\partial \tilde u}{\partial v} & \frac{\partial\tilde v}{\partial v}\end{pmatrix}\begin{pmatrix}\tilde\sigma_{\tilde u}\\ \tilde\sigma_{\tilde v}\end{pmatrix}$$

$$\begin{pmatrix}\tilde\sigma_{\tilde u}\\ \tilde\sigma_{\tilde v}\end{pmatrix} = D\Phi \begin{pmatrix}\sigma_u\\ \sigma_v\end{pmatrix}$$

$$\tilde\sigma_{\tilde u}\times\tilde\sigma_{\tilde v} = \det(D\Phi) \quad \sigma_u\times\sigma_v$$

$$N_{\tilde\sigma} = \left(\mathrm{Sign}(\det D\Phi)\right) N_\sigma .$$

<u>Def</u> A surface is orientable if there is an atlas where all transitions have $\det(D\Phi) > 0$.

Thm ~~The~~ S is orientable iff there is a consistent choice of unit normal throughout S.

Pf If orientable, pick atlas with

$\Rightarrow$ $\det(D\Phi) > 0$, let $N_\sigma = \dfrac{\sigma_u \times \sigma_v}{|\sigma_u \times \sigma_v|}$

$\Leftarrow$ On each patch, either $N = \dfrac{\sigma_u \times \sigma_v}{|\sigma_u \times \sigma_v|}$,

or $N = -\dfrac{\sigma_u \times \sigma_v}{|\sigma_u \times \sigma_v|}$. In the latter case,

switch to (v, u) coordinates.