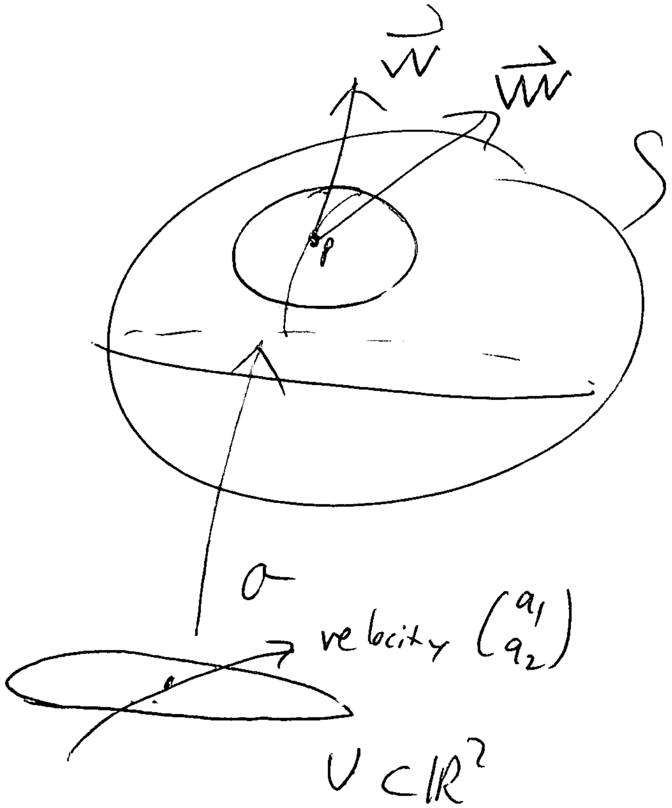


$p = \sigma(u_0, v_0)$

Basis for $T_p S = \{\sigma_u(u_0, v_0),$
$\sigma_v(u_0, v_0)\}$

$$\vec{W} = a_1 \vec{\sigma}_u + a_2 \vec{\sigma}_v$$
$$\vec{WW} = b_1 \vec{\sigma}_u + b_2 \vec{\sigma}_v$$

$du(\vec{w}) = a_1$

$dv(\vec{w}) = a_2$

First fundamental form = ~~i~~ inner product restricted to $T_p S$.

$$\vec{W} \cdot \vec{WW} = a_1 b_1 (\vec{\sigma}_u \cdot \vec{\sigma}_u) + (a_1 b_2 + a_2 b_1) \vec{\sigma}_u \cdot \vec{\sigma}_v + (a_2 b_2) \vec{\sigma}_v \cdot \vec{\sigma}_v$$

$$= (a_1 \ a_2) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$E = \vec{\sigma}_u \cdot \vec{\sigma}_u$

$F = \vec{\sigma}_u \cdot \vec{\sigma}_v$

$G = \vec{\sigma}_v \cdot \vec{\sigma}_v$

$$
\begin{aligned}
\vec{w} \cdot \vec{m} = \; & E\,(du(\vec{w})\,du(\vec{m})) + \\
& + F\,(du(\vec{w})\,dv(\vec{m})) \\
& + F\,(dv(\vec{w})\,du(\vec{m})) \\
& + G\,(dv(\vec{w})\,dv(\vec{m}))
\end{aligned}
$$

$\langle \vec{w} \,|\, \vec{m} \rangle$ or $\langle \vec{w}, \vec{m} \rangle$

$$
\begin{aligned}
\langle \; | \; \rangle = \; & E\,du\,du + F\,du\,dv \\
& + F\,dv\,du + G\,dv\,dv
\end{aligned}
$$

$$
|\vec{w}|^2 = (E\,du^2 + 2F\,du\,dv + G\,dv^2)(\vec{w})
$$

Ex 1: Plane $= \text{Span}\{\vec{x},\vec{y}\}$.

$$\sigma(u,v) = u\vec{x} + v\vec{y}.$$

$$\sigma_u = \vec{x}$$

$$\sigma_v = \vec{y}$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \vec{x}\cdot\vec{x} & \vec{x}\cdot\vec{y} \\ \vec{x}\cdot\vec{y} & \vec{y}\cdot\vec{y} \end{pmatrix} = \text{constant}.$$

Ex 2: Plane w/ polar coordinates.

$$\sigma(\overset{r}{u},\overset{\theta}{v}) = (u\cos v, u\sin v, 0)$$

$$\sigma_u = (\cos v, \sin v, 0) \qquad E = 1 \quad F = 0$$

$$\sigma_v = (-u\sin v, u\cos v, 0) \qquad G = u^2$$

$$ds^2 = \langle \, I \, \rangle = (du)^2 + u^2(dv)^2 = dr^2 + r^2 d\theta^2$$

Ex 3: Sphere $\quad \sigma_{\cancel{u,v}}(u,v) = (\sin u\cos v, \sin u\sin v, \cos u)$

$$\sigma_u = (\cos u\cos v, \cos u\sin v, -\sin u) \qquad E = 1 \quad F = 0$$

$$\sigma_v = (-\sin u\sin v, \sin u\cos v, 0) \qquad F = 0 \quad G = \sin^2 u$$

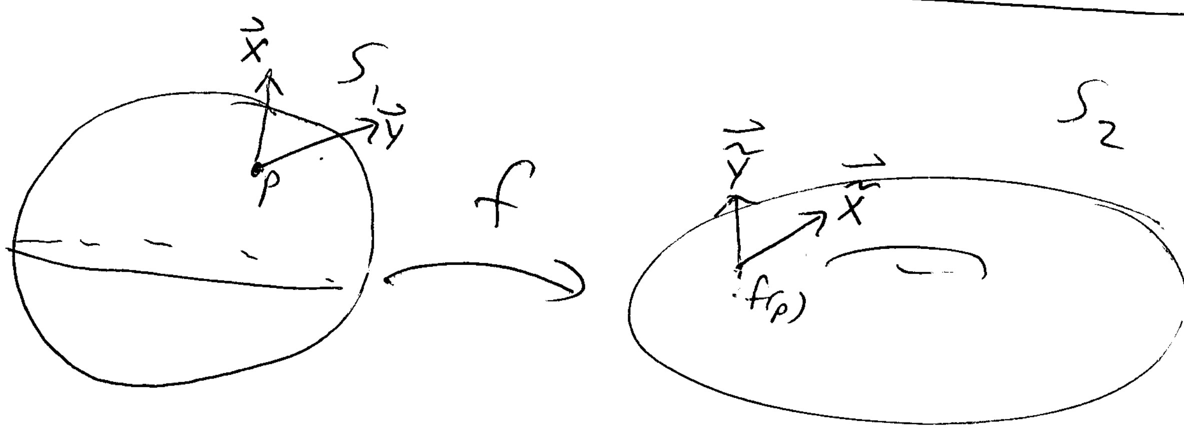
$$ds^2 = (du)^2 + \sin^2 u\,(dv)^2$$

Ex 4

$$\sigma(u,v) = (\cancel{a}\cos u, \cancel{\cos}\sin u, v)$$

$$\sigma_u = (-\sin u, \cos u, 0) \qquad E = 1 \qquad F = 0$$

$$\sigma_v = (0, 0, 1) \qquad F = 0 \qquad G = 1$$



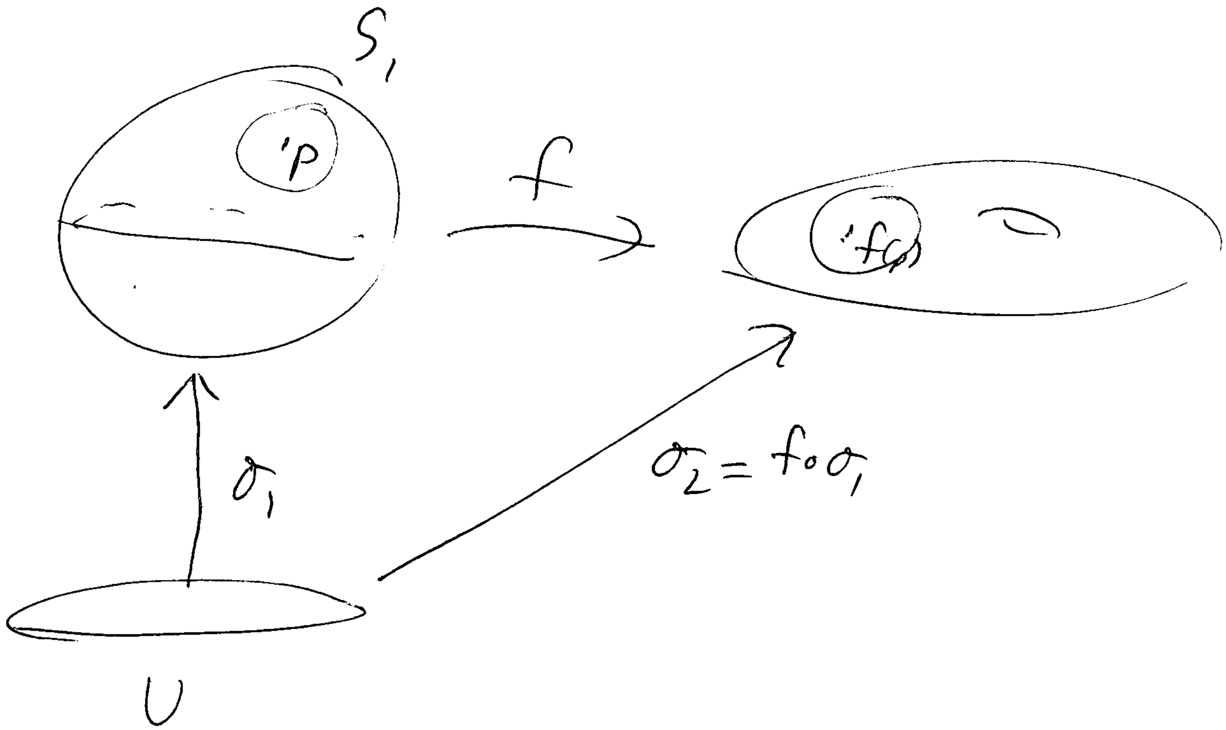
Def

f is ~~an~~ a <u>local isometry</u> if for any curve γ on S_1, $f \circ \gamma$ has same length as γ.

$$\tilde{\vec{X}} = Df(\vec{X})$$
$$\tilde{\vec{Y}} = Df(\vec{Y})$$

~~Df~~ Def $\underbrace{f^* \langle \, , \rangle}$

$$(f^* \langle \, , \rangle)(\vec{X}, \vec{Y}) := \langle \tilde{X}, \tilde{Y} \rangle_{f(p)}$$



<u>Thm</u> The following are equivalent.

1) f is a local isometry near p.
2) $f^*\langle\ ,\ \rangle_{S_2} = \langle\ ,\ \rangle_{S_1}$
3) σ_1 and σ_2 have the same $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$

Pf: (1) $\iff$ $f\circ\gamma$ has same speed as γ

$\iff |Df(x)| = |x|$ for all $x \in TS_1$

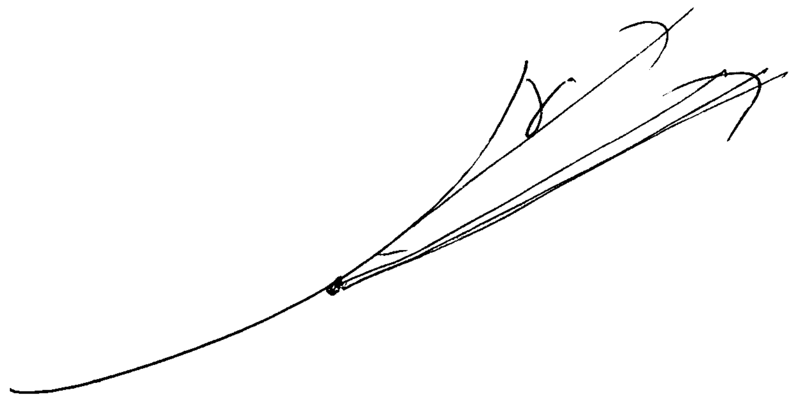
$\iff f^*\langle\ ,\ \rangle(x,x) = \langle x,x\rangle$ for all $x \in TS_1$

But $\langle x,y\rangle = (|x+y|^2 - |x-y|^2)/4$

$\iff$ (2)

But $\cancel{f}\ Df\,(\sigma_1)_u = (\sigma_2)_u$,
$Df\,(\sigma_1)_v = (\sigma_2)_v$

Suppose $\gamma =$ curve $, |\dot{\gamma}| = 1, \ \ddot{\gamma} \neq 0.$



$$\sigma(u,v) = \gamma(u) + v\,\dot{\gamma}(u)$$

$$\sigma_u = \dot{\gamma} + v\,\ddot{\gamma}$$

$$\sigma_v = \dot{\gamma}$$

$\sigma_u \times \sigma_v = v\,\ddot{\gamma} \times \dot{\gamma} \neq 0$
when $v > 0$.

Claim Tangent developable = locally isometric to piece of $\mathbb{R}^2$.

pf
$$E = 1 + v^2 \kappa^2(u)$$
$$F = 1$$
$$G = 1$$

$$\begin{pmatrix} 1+v^2\kappa^2 & 1 \\ 1 & 1 \end{pmatrix}$$

Depends only on κ, not on τ.
So surface is locally isometric to tangent developable of plane curve.

Inner products measure

1) length
2) angles,
3) area.

Preserving angle $\Longleftrightarrow$ conformal

Preserving area $\Longleftrightarrow$ equaareal (sp?)

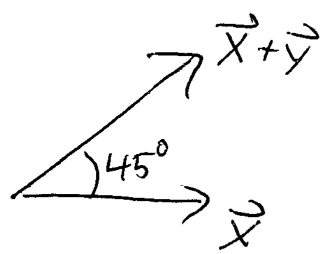
Claim: Angles define $\langle , \rangle$ up to scale.

Suppose $\vec{x}, \vec{y}$ were orthonormal. In new $\langle , \rangle$,

$$\langle \vec{x}, \vec{x} \rangle = c_1^2$$

$$\langle \vec{x}, \vec{y} \rangle = 0$$

$$\langle \vec{y}, \vec{y} \rangle = c_2^2$$



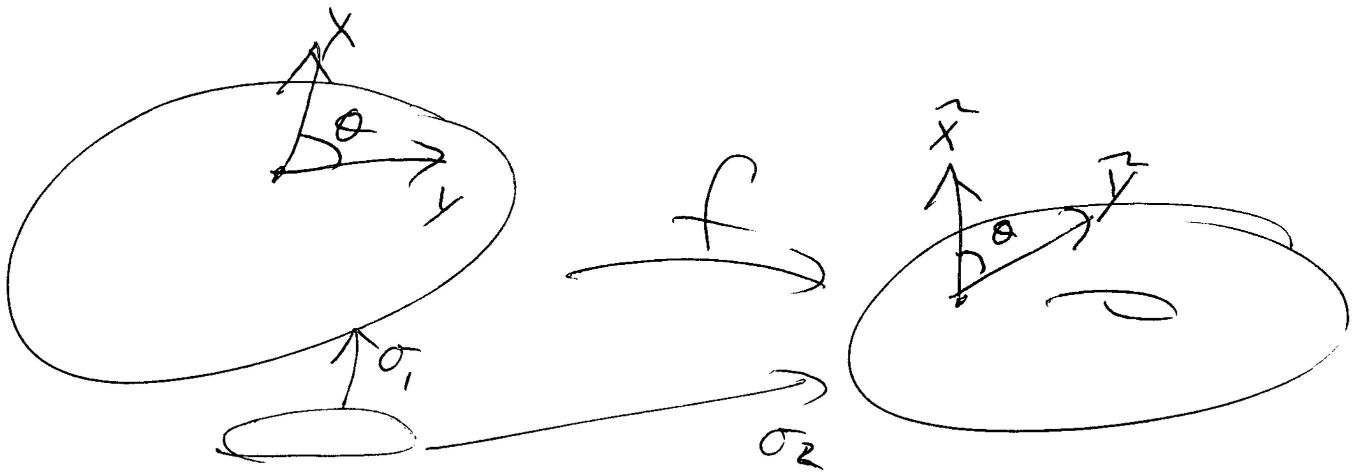
$$\langle \vec{x}+\vec{y}, \vec{x} \rangle = |x+y| \, |x| \, \frac{\sqrt{2}}{2}$$

$$c_1^2 = \sqrt{c_1^2+c_2^2} \, c_1 \, \frac{\sqrt{2}}{2}$$

$$\sqrt{c_1^2+c_2^2} = \sqrt{2} \, c_1 \implies c_1^2+c_2^2 = 2c_1^2$$

$$c_1^2 = c_2^2$$

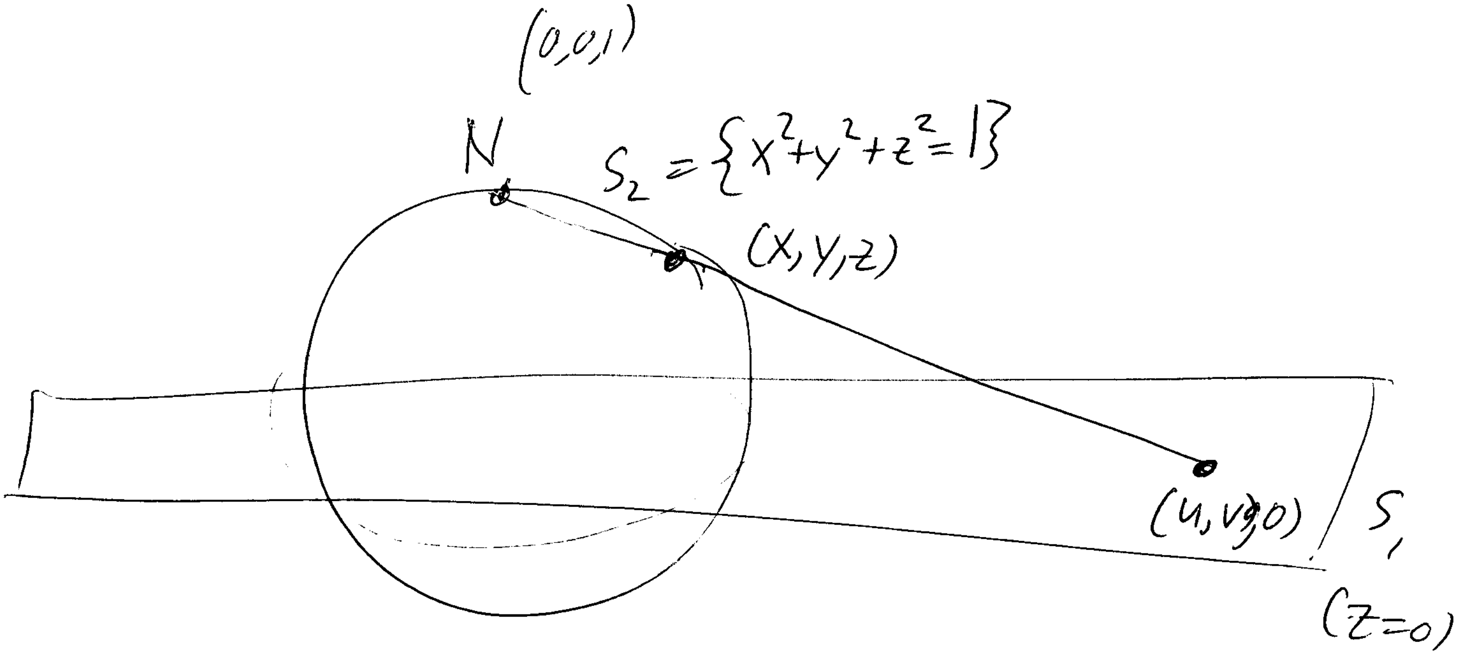
A map between surfaces is <u>conformal</u> if it preserves angles



$$\Longleftrightarrow \quad f^*\langle \, , \rangle = c^2 \langle \, , \rangle$$

$$\Longleftrightarrow \quad \begin{pmatrix} E_2 & F_2 \\ F_2 & G_2 \end{pmatrix} = c^2 \begin{pmatrix} E_1 & F_1 \\ F_1 & G_1 \end{pmatrix}$$

Stereographic Projection.



$$\sigma(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$$

$$\sigma_u = \frac{1}{(u^2+v^2+1)^2}\left(2+2v^2-2u^2, -4uv, 4u\right)$$

$$\sigma_v = \frac{1}{(u^2+v^2+1)^2}\left(-4uv, 2+2u^2-2v^2, 4v\right)$$

$$G = \frac{1}{(u^2+v^2+1)^4}\left(16u^2v^2 + 16v^2 + (2+2u^2-2v^2)^2\right) = \frac{4}{(u^2+v^2+1)^2}$$

E = Similar

$$F = \frac{-8uv - 8uv^3 + 8u^3v - 8uv - 8u^3v + 8uv^3 + 16uv}{(\quad)^4} = 0$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \frac{4}{(u^2+v^2+1)^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$