Whats a smooth Surface?

Something that locally looks like $\mathbb{R}^2$.

1) Graphs of smooth functions $f : \mathbb{R}^2 \to \mathbb{R}$.

$(x, y, f(x,y))$ or $(x, f(x,z), z)$ or $(f(y,z), y, z)$.

2) Image of smooth $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$ with Rank $D\sigma = 2 \iff \sigma_u \times \sigma_v \neq 0$.

3) Level set of smooth $F : \mathbb{R}^3 \to \mathbb{R}$. with $\nabla F \neq 0$ on level set,

(Local) Equivalence via IFT.

Unit normal vector $= \pm \dfrac{\sigma_u \times \sigma_v}{|\sigma_u \times \sigma_v|}$

S is orientable if consistent choice.

Level sets are orientable.

(Pick $\vec{N}$ in ∇F direction).

Examples.

Quadrics: level sets of $F(x) = x^T A x + b^T x$.

Shape depends mostly on e-vals of A.

$(+,+,+)$: ellipsoid (or pt. or $\emptyset$)
$(-,-,-)$

$\left.\begin{matrix}(+,+,-)\\(-,-,+)\end{matrix}\right\}$ hyperboloid of 1 or 2 sheets or double cone.

Paraboloids, cylinders, planes, lines, pts, occur when e-val(s) $= 0$.

Ruled surfaces = collections of lines.

$$\sigma(u,v) = \vec{\gamma}(u) + v\,\vec{\delta}(u)$$

$$D\sigma = \begin{pmatrix} \dot{\gamma} + v\dot{\delta} \\ \delta \end{pmatrix}$$

E.g. $\quad z = xy \qquad \left.\begin{array}{l} \gamma(u) = (u,0,0) \\ \delta(u) = (0,1,u) \end{array}\right\} \sigma(u,v) = (u,v,uv)$

$x^2+y^2-z^2=1 \qquad \begin{array}{l} \gamma(u) = (\cos u,\ \sin(u),\ 0) \\ \delta(u) = (-\sin u,\ \cos(u),\ \pm 1) \end{array}$

Generalized cylinder $(\delta = \text{const})$

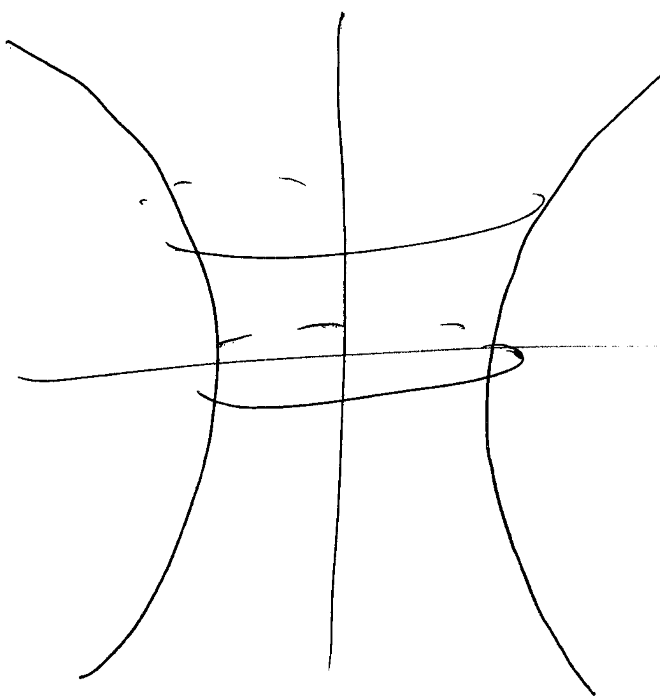
" $\quad$ Cone $\quad (\delta = \gamma)$

Surfaces of revolution

$$\gamma(u) = (f(u), 0, g(u))$$

$$\sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$$

Ex: $f(u) = \cosh u, \quad g(u) = \sinh(u)$



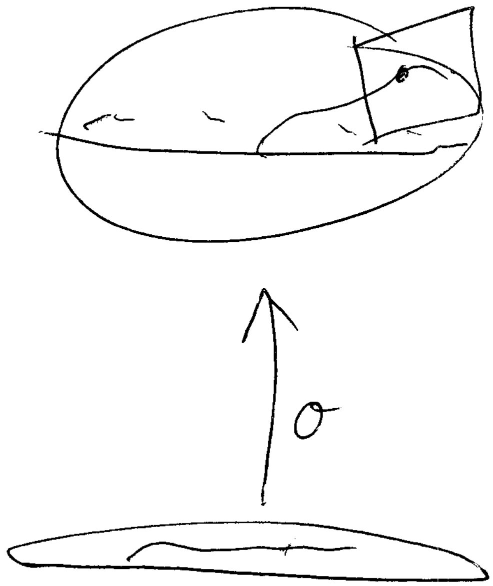
hyperboloid of 1 sheet.

Compact surfaces

Classification: $S = S^2, T^2,$ or ~~#~~

g-holed torus.

All are orientable.



$T_pS = \{\text{all possible velocities through } p\}$

$= \text{Span}\{\sigma_u, \sigma_v\} \subset \mathbb{R}^3$

First Fundamental Form (FFF) is inner product restricted to T_pS.

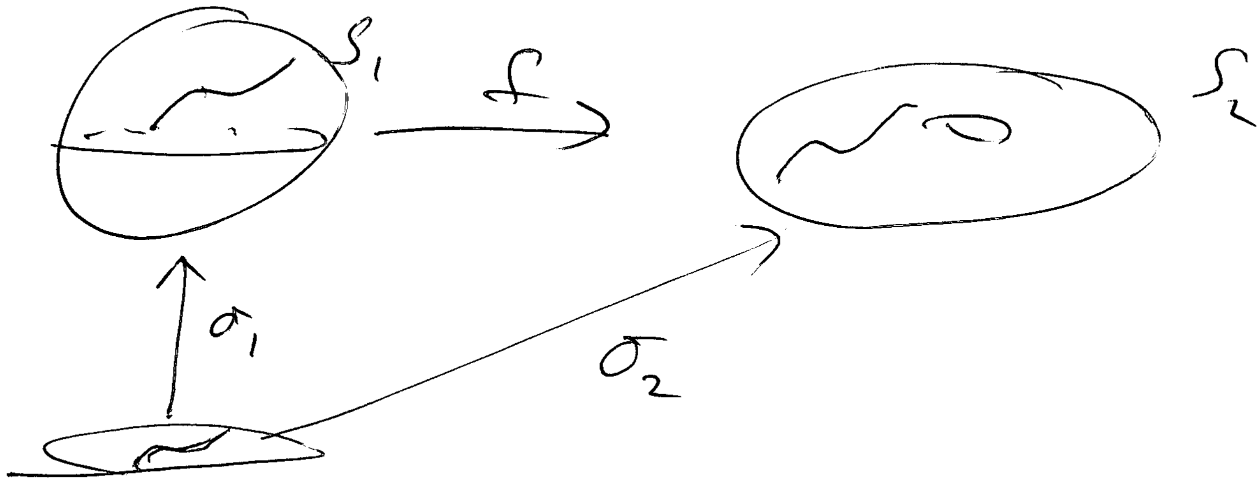
Work in σ_u, σ_v basis,

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \sigma_u \cdot \sigma_u & \sigma_u \cdot \sigma_v \\ \sigma_v \cdot \sigma_u & \sigma_v \cdot \sigma_v \end{pmatrix}$$

$$ds^2 = E(du)^2 + 2F\,du\,dv + G(dv)^2$$

$$\text{Length} = \int \sqrt{E(\dot{u})^2 + 2F\dot{u}\dot{v} + G(\dot{v})^2}\,dt$$

Isometries preserve FFF.



<u>Thm</u> f is a local isometry if, for any chart σ_1, $\begin{pmatrix} E_2 & F_2 \\ F_2 & G_2 \end{pmatrix} = \begin{pmatrix} E_1 & F_1 \\ F_1 & G_1 \end{pmatrix}$, where

$\sigma_2 = f \circ \sigma_1$

Sphere: ~~$\sigma(u,v) =$~~

$$\sigma(\theta,\phi) = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$$

$$ds^2 = (d\theta)^2 + \cos^2\theta\,(d\phi)^2$$

$$FFF = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2\theta \end{pmatrix}$$

Stereographic coordinates

$$\sigma(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$$

$$d\sigma^2 = (du^2+dv^2)\cdot\frac{4}{(u^2+v^2+1)^2}$$

Conformal maps preserve angle.

<u>Thm</u>

f is conformal if

$$\begin{pmatrix} E_2 & F_2 \\ F_2 & G_2 \end{pmatrix} = h(u,v) \begin{pmatrix} E_1 & F_1 \\ F_1 & G_1 \end{pmatrix}$$

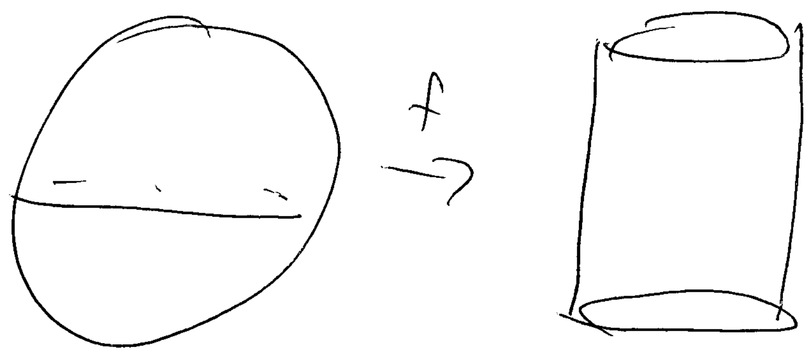
(<u>Deep fact</u> All surfaces have (locally) conformal coordinates.)

When does a map preserve area?

Ans: When $E_2 G_2 - F_2^2 = E_1 G_1 - F_1^2$.

$$\text{Area} = \iint \sqrt{EG - F^2}\, du\, dv$$

Application



$$(x, y, z) \longrightarrow \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, z \right)$$

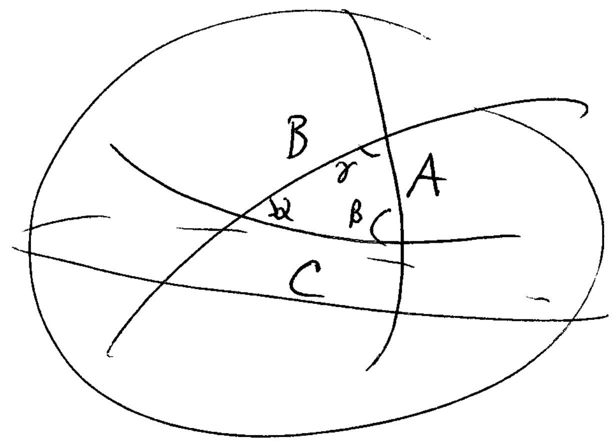
$$ds_1^2 = \cancel{\cos}\, d\theta^2 + \cos^2\theta\, d\phi^2$$

$$ds_2^2 = \cancel{\phi}\, \cos^2\theta\, d\theta^2 + d\phi^2$$
$$= (dz)^2 + (d\phi)^2$$

$z = \sin\theta$

$dz = \cos\theta\, d\theta$

Spherical geometry:

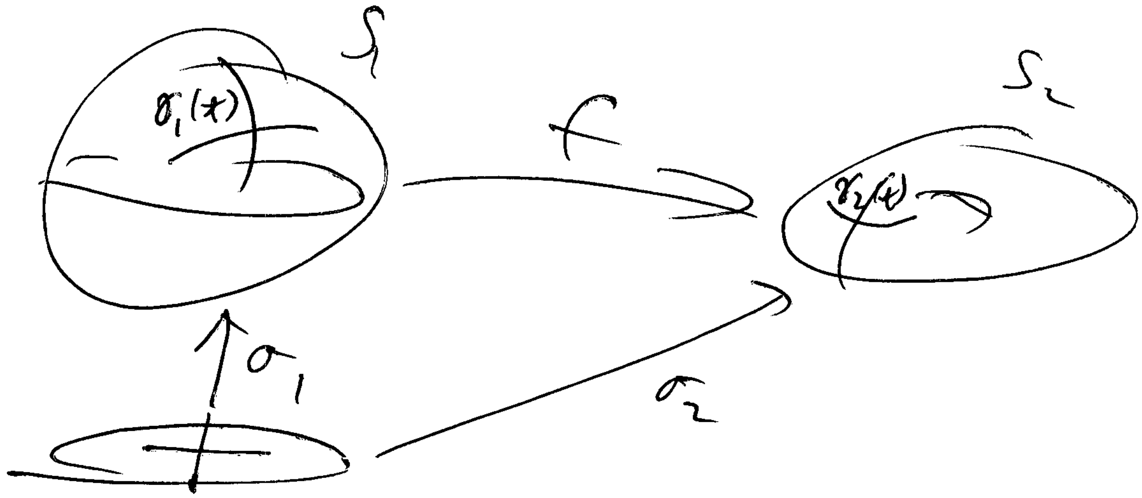


$$\alpha + \beta + \gamma = \pi + \text{Area}\,\Delta$$

$$\cos C = \cos A \cos B + \sin A \sin B \cos \gamma$$

$$\frac{\sin \alpha}{\sin A} = \frac{\sin \beta}{\sin B} = \frac{\sin \gamma}{\sin C}$$

Suppose f is local isometry.



Consider path $(u,v) = (t,0)$

Speed of γ_1 at $t=0$ is $|\sigma_u| = \sqrt{E_1}$.

Speed of γ_2 is $\sqrt{E_2}$. So $\sqrt{E_1} = \sqrt{E_2}$, so $E_1 = E_2$.

Now consider path $(u,v) = (0,t)$ $\qquad G_1 = G_2$

Now consider $(u,v) = (t,t)$

$$\sqrt{E_1+G_1+2F_1} = \sqrt{E_2+G_2+2F_2} \qquad (\dot u \ \dot v)\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \dot u \\ \dot v \end{pmatrix}$$

$$\int \sqrt{E_1 \dot u^2 + G_1 \dot v^2 + 2F_1 \dot u \dot v}\, dt \quad \text{vs} \ldots$$