$V =$ vector space. (dim $= n$)

Basis $\vec{e}_1, \vec{e}_2, \ldots \vec{e}_n$

$$\vec{V} = \sum_i V^i \vec{e}_i = V^1 \vec{e}_1 + \cdots + V^n \vec{e}_n$$

Different basis $\tilde{\vec{e}}_1, \ldots \tilde{\vec{e}}_n$

Change-of basis. $\tilde{e}_i = \sum_j A_i^{\,j} e_j$

$$\vec{V} = \tilde{V}^1 \tilde{e}_1 + \cdots \tilde{V}^n \tilde{e}_n$$

$$= \sum_j \tilde{V}^1 A_1^{\,j} e_j + \tilde{V}^2 A_2^{\,j} e_j + \cdots \tilde{V}^n A_n^{\,j} e_j$$

$$= \sum_{j,K} \tilde{V}^K A_K^{\,j} e_j = \sum_j V^j e_j$$

$$V^j = \sum_K A_K^{\,j} \tilde{V}^K$$

$$\alpha \in V^*$$

$$\alpha = \sum_i \alpha_i \phi^i$$

$$\alpha(v) = \sum_i \alpha_i \phi^i(v) = \sum_i \alpha_i v^i$$

$$\alpha(e_i) = \alpha_i$$

Vector space V

Basis $\{e_1, \dots e_n\}$

vector $\vec{v} = \sum_i v^i e_i$

$$\phi^i(v) = v^i$$

dual space V^*

dual basis $\{\phi^1, \dots \phi^n\}$

$$\phi^i(e_j) = \delta^i_j$$

Co-vector $\alpha = \sum_j \alpha_j \phi^j$

$$\cancel{\phi^i(v)} \quad \alpha(e_j) = \alpha_j$$

lower indices transform by A

upper by $B = A^{-1}$

$B = A^{-1}$ $\quad \sum_j B_i^{\ j} A_j^{\ k} = \delta_i^{\ k} = \begin{cases} 1 \text{ if } i=k \\ 0 \text{ if } i \neq k \end{cases}$

$$\sum_k A_j^{\ k} B_k^{\ l} = \delta_j^{\ l}$$

$$\tilde{v}^j = \sum_k B_k^{\ j} v^k$$

Dual space: $V^* = \hom(V, \mathbb{R})$

Dual basis $\phi^1, \ldots \phi^{\cancel{k}n}$

$\phi^i(e_j) = \delta_j^i$ $\qquad$ Ex: $V = \mathbb{R}^2$,

$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\phi_1 = (1\ 0), \; \phi_2 = (0, 1)$

$$\phi^i(\vec{v}) = \phi^i\left(\sum_j v^j e_j\right) = \sum_j v^j \phi^i(e_j) = \sum_j v^j \delta_j^i = v^i$$

Change of basis

$$\tilde{\phi}^j(\vec{v}) = \tilde{v}^j = \sum_k B_k^{\ j} v^k = \sum_k B_k^{\ j} \phi^k(\vec{v})$$

$$\tilde{\phi}^j = \sum_k B_k^{\ j} \phi^k$$

~~∅~~ In $\mathbb{R}^n$,

$\vec{v}$ is a vector

$(\vec{v} \cdot \quad)$ is a covector.

$f : \mathbb{R}^3 \to \mathbb{R}$.

$df = (\partial_1 f, \partial_2 f, \partial_3 f) = (\nabla f \cdot \quad)$

A 2-tensor is a machine that takes 2 objects $\rightarrow$ #

+ is Linear in each object.

Covariant (2,0) tensors.

Take 2 vectors, give #.

$$T(v,w) = \sum_{ij} T(v^i e_i, w^j e_j)$$

$$= \sum_{ij} v^i w^j \, T(e_i, e_j) = \sum_{ij} T_{ij} v^i w^j$$

$$\tilde{T}_{ij} = T(\hat{e}_i, \tilde{e}_j) = \sum_{i',j'} A_i^{i'} A_j^{j'} T_{i'j'}$$

$$\text{“}=\text{”} \; A^T T A$$

(1,1) tensor takes 1 vector & 1 covector & gives #.

$$\cancel{F(v,\alpha)=}\; T(\alpha, v) = \sum_{ij} \alpha_i T^i_j v^j$$

where $T^i_j = T(\phi^i, e_j)$

$$\begin{pmatrix} \alpha \end{pmatrix} \begin{pmatrix} \\ T \\ \\ \end{pmatrix} \begin{pmatrix} \\ v \\ \\ \end{pmatrix}$$

$$\tilde{T}^i_j = \sum_{i'j'} A^{j'}_j B^i_{i'} T^{i'}_{j'}$$

$$= A^{-1} T A = B T B^{-1}$$

$\langle\, , \,\rangle$ is $(2,0)$ tensor.

$\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ is matrix in σ_u, σ_v basis.

$E = \langle \sigma_u, \sigma_u \rangle = \sigma_u \cdot \sigma_u$

$F = \langle \sigma_u, \sigma_v \rangle = \langle \sigma_v, \sigma_u \rangle = \sigma_u \cdot \sigma_v$

$G = \langle \sigma_v, \sigma_v \rangle = \sigma_v \cdot \sigma_v$

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} T(e_1, e_1) & T(e_1, e_2) \\ T(e_2, e_1) & T(e_2, e_2) \end{pmatrix}$$

$\langle\langle v, w \rangle\rangle = -\vec{N}_v \cdot \vec{w}$

$L = \langle\langle \sigma_u, \sigma_u \rangle\rangle = -\vec{N}_u \cdot \sigma_u = \vec{N} \cdot \vec{\sigma}_{uu}$

$M = \langle\langle \sigma_u, \sigma_v \rangle\rangle = -\vec{N}_u \cdot \vec{\sigma}_v = \vec{N} \cdot \vec{\sigma}_{uv}$

$M = \langle\langle \sigma_v, \sigma_u \rangle\rangle = -\vec{N}_v \cdot \vec{\sigma}_u = \vec{N} \cdot \vec{\sigma}_{vu}$

$N = \qquad -\vec{N}_v \cdot \vec{\sigma}_v = \vec{N} \cdot \vec{\sigma}_{vv}$

$\begin{pmatrix} L & M \\ M & N \end{pmatrix}$

Weingarten map (aka Shape operator)

$$\vec{v} \longrightarrow -\vec{N}_{\vec{v}}$$

$$\mathcal{W}(\vec{v}) = -N_{\vec{v}}$$

$$\langle\langle \vec{v}, \vec{w} \rangle\rangle = \mathcal{W}(\vec{v}) \cdot \vec{w}$$

If $\vec{v} = c_1 \vec{\sigma}_u + c_2 \vec{\sigma}_v$

$$\vec{N}_{\vec{v}} := c_1 \vec{N}_u + c_2 \vec{N}_v$$

matrix of $\mathcal{W} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}$

$$\det \mathcal{W} = \frac{LN - M^2}{EG - F^2}$$

eigenvalues of W = principal curvatures, κ_1, κ_2

$\frac{1}{2} \operatorname{Tr} W$ = mean curvature $= \dfrac{\kappa_1 + \kappa_2}{2} = H$

$\det W$ = Gaussian curvature $= \kappa_1 \kappa_2 = K$