$A : \mathbb{R}^n \to \mathbb{R}^n$

When does A preserve

1) inner product?
2) angles?
3) volume?

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(1) Want $(A\vec{x}) \cdot (A\vec{y}) = \vec{x} \cdot \vec{y}$

$$= \vec{x}^T \vec{y}$$

$$= (x_1, \dots x_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

Want $x^T A^T A y = x^T y$ for all $\vec{x}, \vec{y}$.

Want $A^T A = I$

If $A = (\vec{a}_1, \dots \vec{a}_n)$ $A^T = \begin{pmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_n^T \end{pmatrix}$

$$A^T A = \begin{pmatrix} \vec{a}_1 \cdot \vec{a}_1 & \vec{a}_1 \cdot \vec{a}_2 & \cdots & \vec{a}_1 \cdot \vec{a}_n \\ \vdots & & & \\ & & & \\ & & & \vec{a}_n \cdot \vec{a}_n \end{pmatrix}$$

A preserves $\langle\,,\rangle \iff$ Columns of A are orthonormal

$\iff A^T A = I$

$\iff A A^T = I$

$\iff$ Columns of A^T are orthonormal

$\iff A$ preserves length.

$$\left(\vec{x}\cdot\vec{y} = \left(|\vec{x}+\vec{y}|^2 - |\vec{x}-\vec{y}|^2\right)/4\right)$$

~~Ef~~ ~~Df~~

<u>Def</u>: An orthogonal matrix is a square matrix A with $A^T A = I$.

<u>Fact</u>: Orthogonal matrices form a group $O(n)$

$$1 = \operatorname{Det} I = \operatorname{Det}(A^T A) = (\operatorname{Det} A^T)(\operatorname{Det} A) = (\operatorname{Det} A)^2$$

$$\operatorname{Det} A = \underset{\text{rotation}}{+1} \quad \text{or} \quad \underset{\text{reflection \& rotation.}}{-1}$$

$SO(n)$ = orthogonal matrices w/ Det $= +1$.

What if we just care about angles?

Then need $A^T A = c^2 I$.

Columns (or rows) of A are orthogonal and have the same size

$A = c \cdot$ (orthogonal matrix)

What if you just care about volume?

Want $\det A = 1$

$\mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2$ is conformal when

Df preserves angles

$\Rightarrow Df = c^2 \cdot \text{rotation}.$ (or reflection).

$$Df = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

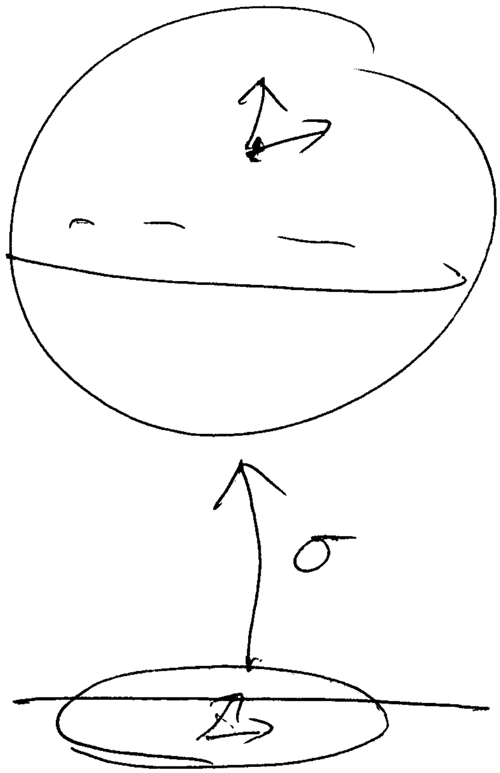
$\det = a^2 + b^2 \qquad \det = -a^2 - b^2 < 0$

$\mathbb{C} \xrightarrow{f} \mathbb{C} \qquad w = f(z)$

$$\begin{pmatrix} \frac{\partial \operatorname{Re} w}{\partial \operatorname{Re} z} & \frac{\partial \operatorname{Re} w}{\partial \operatorname{Im} z} \\ \frac{\partial \operatorname{Im} w}{\partial \operatorname{Re} z} & \frac{\partial \operatorname{Im} w}{\partial \operatorname{Im} z} \end{pmatrix}$$

Cauchy-Riemann: $\dfrac{\partial \operatorname{Re} w}{\partial \operatorname{Re} z} = \dfrac{\partial \operatorname{Im} w}{\partial \operatorname{Im} z}$

$$\frac{\partial \operatorname{Im} w}{\partial \operatorname{Re} z} = -\frac{\partial \operatorname{Re} w}{\partial \operatorname{Im} z}$$



<u>Def</u>

σ is a conformal parametrization if angle on S = angles in $\mathbb{R}^2$

σ is conformal if $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ = mutiple of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\iff E=G$ and $F=0$.

$\iff \sigma_u \cdot \sigma_u = \sigma_v \cdot \sigma_v$ and $\sigma_u \cdot \sigma_v = 0$

$\iff \sigma_u$ and σ_v are $\perp$ and the same size.

$\sigma : \mathbb{R}^2 \to S^2$

$$\sigma(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{u^2+v^2-1}{1+u^2+v^2} \right)$$

$$E = \overset{G}{\cancel{F}} = \frac{4}{(1+u^2+v^2)^2}, \quad F = 0$$

Claim: $\text{Area} = \iint |\sigma_u \times \sigma_v| \, du\, dv$

$$= \iint \sqrt{EG - F^2} \, du\, dv$$

$|\sigma_u \times \sigma_v| =$

pf: $\sqrt{(\sigma_u \times \sigma_v) \cdot (\sigma_u \times \sigma_v)} = \sqrt{(\sigma_u \cdot \sigma_u)(\sigma_v \cdot \sigma_v) - (\sigma_u \cdot \sigma_v)(\sigma_v \cdot \sigma_u)} = \sqrt{EG - F^2}$

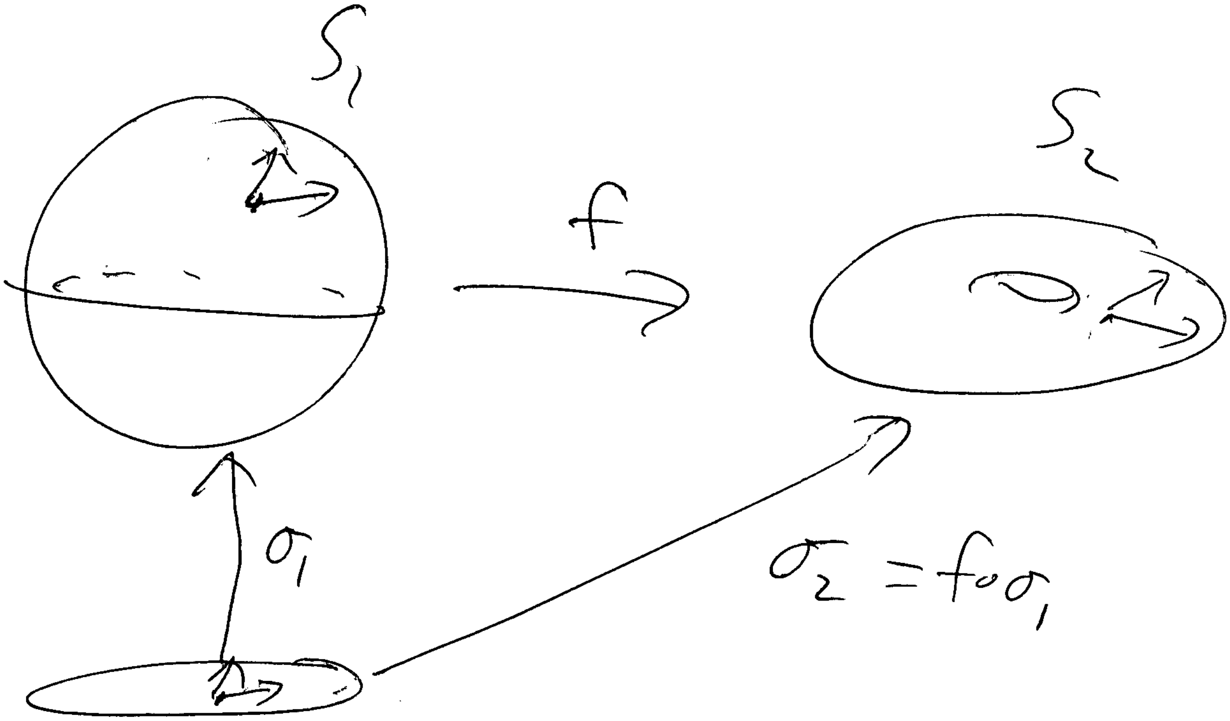
Area of $S^2 = \iint \sqrt{EG-F^2}\, du\, dv$

$$= \iint \frac{4}{(1+u^2+v^2)^2}\, du\, dv$$

$$= \iint \frac{4\, r\, dr\, d\theta}{(1+r^2)^2}$$

$$= 8\pi \int_0^\infty \frac{r}{(1+r^2)^2}\, dr \qquad \begin{array}{l} z = 1+r^2 \\ dz = 2r\,dr \end{array}$$

$$= 8\pi \int_1^\infty \frac{\frac{1}{2} dz}{z^2} = -\frac{4\pi}{z}\Big|_1^\infty = 4\pi$$



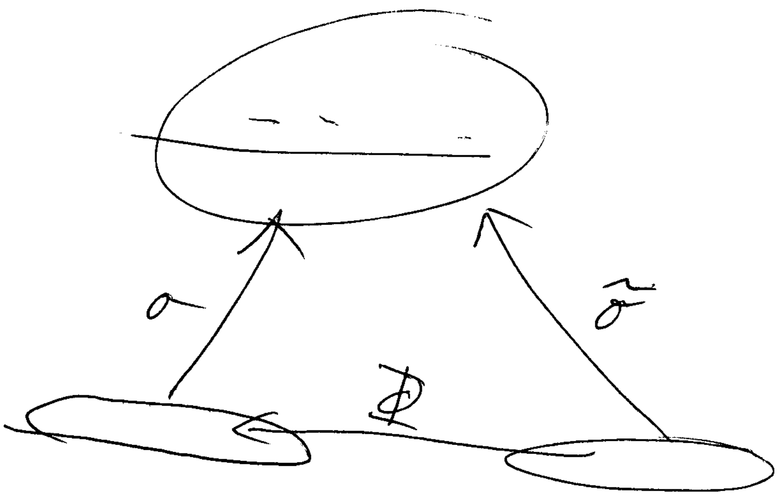
Def f is conformal if f preserves angles.

Fact f is conformal iff $\begin{pmatrix} E_2 & F_2 \\ F_2 & G_2 \end{pmatrix} = c^2 \begin{pmatrix} E_1 & F_1 \\ F_1 & G_1 \end{pmatrix}$

Facts about areas

1) $$\text{Area} = \iint \sqrt{EG-F^2}\, du\,dv.$$
$$= \iint |\sigma_u \times \sigma_v|\, du\,dv.$$

2)



$$\begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} = (D\Phi)^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} (D\Phi)$$

$\cancel{\tilde{E}\tilde{F}}$

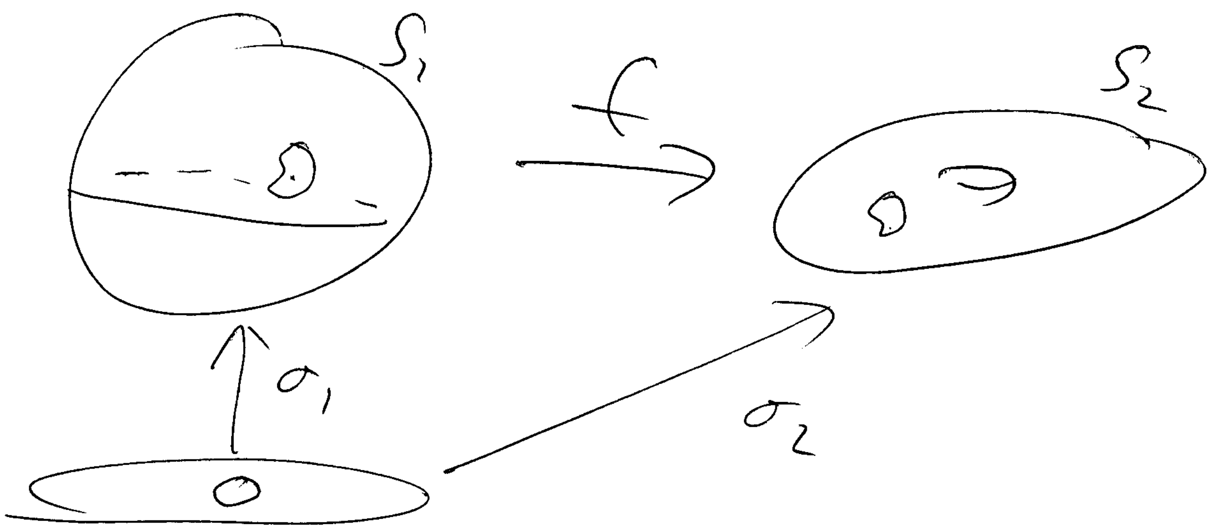
$$\tilde{E}\tilde{G}-\tilde{F}^2 = (\det D\Phi)^2 (EG-F^2)$$

$$\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2} = |\det D\Phi| \sqrt{EG-F^2}$$

$$\iint \sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}\, d\tilde{u}\, d\tilde{v} = \iint \cancel{\sqrt{EG+}}$$

$$= \iint \sqrt{EG-F^2}\, \underbrace{|\det D\Phi|\, d\tilde{u}\, d\tilde{v}}_{du\, dv}$$

~~Ans~~ Area does not depend on choice of coordinates.



f preserves area iff $E_2G_2 - F_2^2 = E_1G_1 - F_1^2$

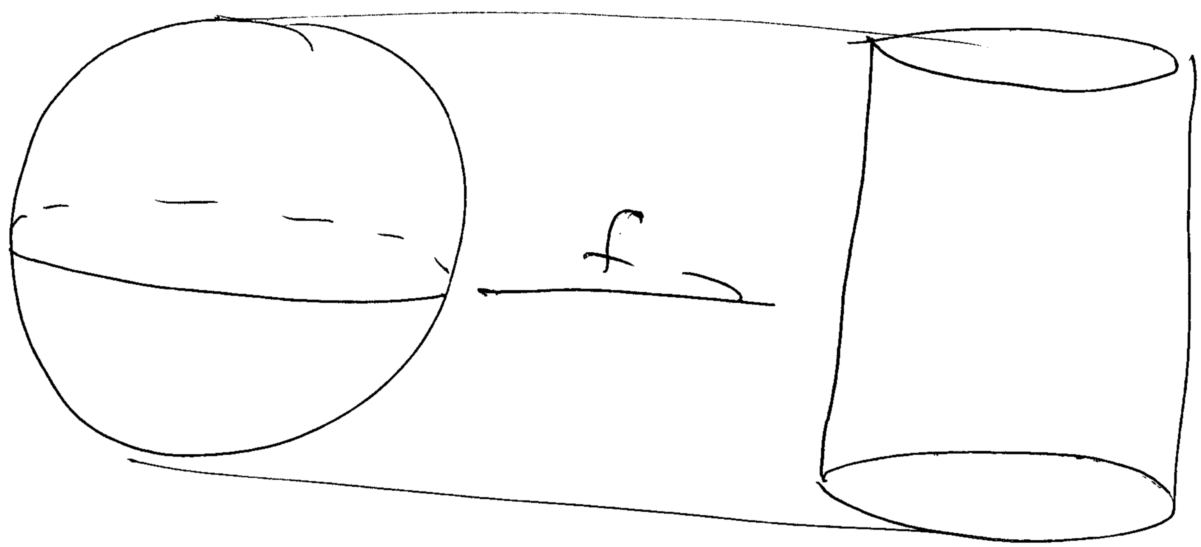
3) $x^2+y^2+z^2=1, \; -1<z<1$

$x^2+y^2=1$

$-1<z<1$

$S_1 =$ 2-sphere (minus poles)

$S_2 =$ cylinder



$$f(x,y,z) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, z\right)$$

$$\sigma_1(\cancel{\theta,\phi} = u,v) = (\cos u \, \cancel{\sin v} \cos v, \; \cos u \, \cancel{\cos v} \sin v, \; \sin u)$$

$$\sigma_2(u,v) = f(\sigma_1(u,v)) = (\cancel{\sin v,}$$

$$(\cos v, \sin v, \sin u)$$

$$(\sigma_1)_u = (-\sin u \cos v, -\sin u \sin v, \cos u) \qquad E_1 = 1 \quad F_1 = 0$$

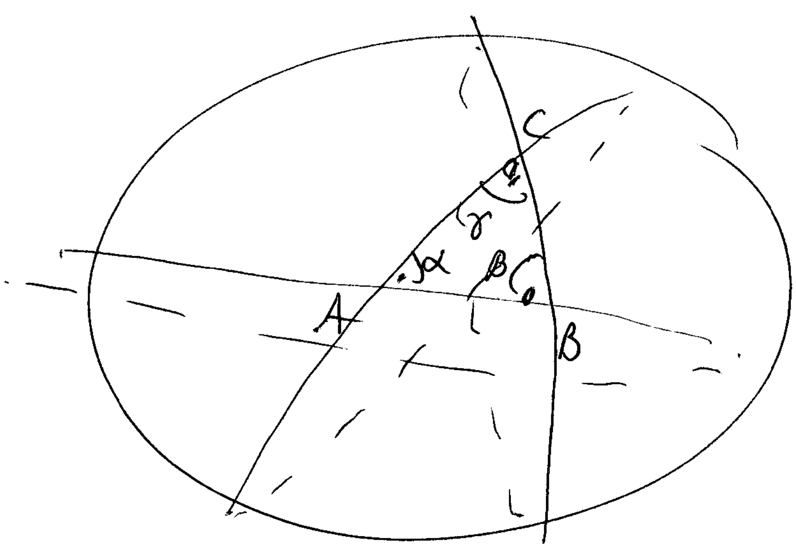
$$(\sigma_1)_v = (\cancel{0} - \cos u \sin v, + \cos u \cos v, 0) \qquad G_1 = \cos^2 u$$

$(\sigma_2)_u = (0, 0, \cos u)$ $\qquad E_2 = \cos^2 u \quad F_2 = 0$

$(\sigma_2)_v = (-\sin v, \cos v, 0)$ $\qquad G_2 = 1$

$$E_1 G_1 - F_1^2 = \cos^2 u = E_2 G_2 - F_2^2$$

Spherical geometry



4)

<u>Thm</u> $\alpha + \beta + \gamma = \pi + \text{Area(triangle } ABC)$