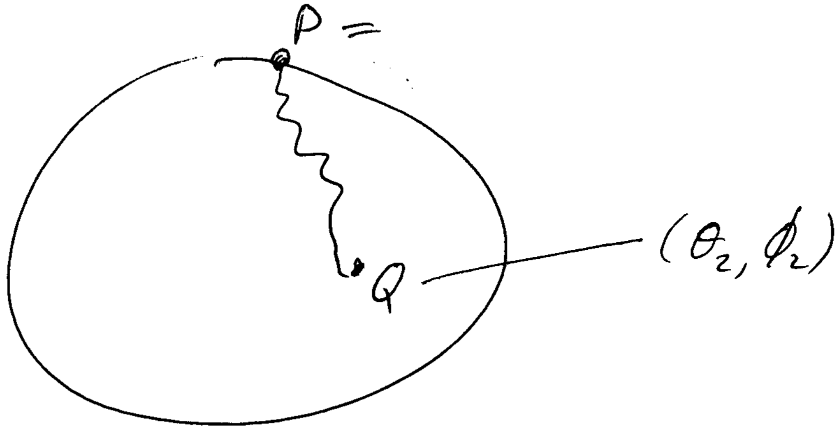
Def: A "line" is a shortest path between two pts.

Thm Lines are great circles.

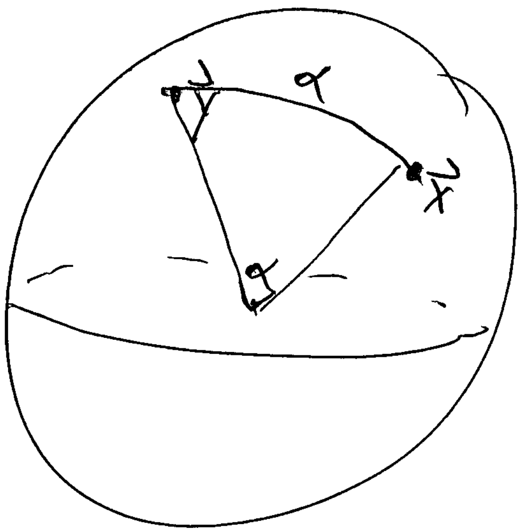


$\gamma(t) = \sigma(\theta(t), \phi(t))$

Length of $\gamma = \int |\dot{\gamma}| dt = \int \sqrt{(\dot{\theta})^2 + \cos^2\theta (\dot{\phi})^2}\, dt$

$\geq \int |\dot{\theta}| dt$ = length of great circle.

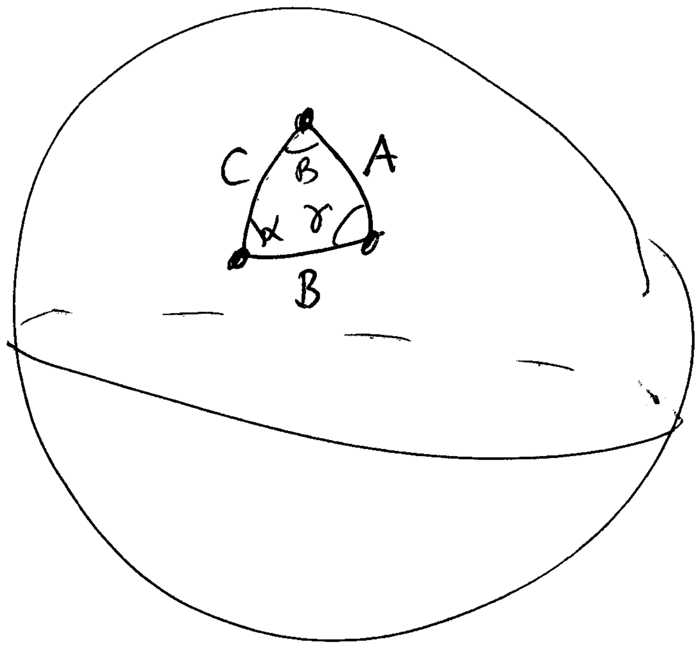
Thm



$$\cos \underbrace{d_{S^2}(x,y)}_{\alpha} = \cos\alpha = \vec{x}\cdot\vec{y}.$$

$$\sin\alpha = |\vec{x}\times\vec{y}|$$

Thm (Law of cosines)



$$\cos C = \cos A \cos B + \sin A \sin B \cos \gamma.$$

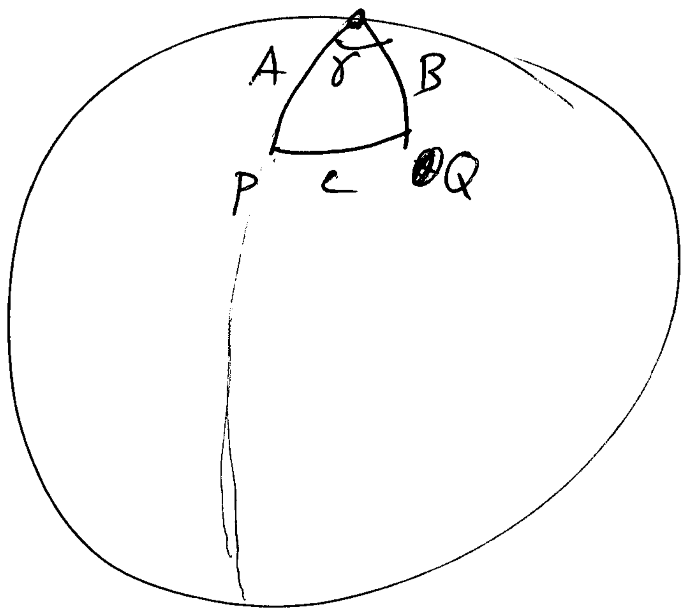
In Euclidean limit,

$$1 - \frac{C^2}{2} = \left(1 - \frac{A^2}{2}\right)\left(1 - \frac{B^2}{2}\right) + AB \cos\gamma$$

$$= 1 - \frac{A^2}{2} - \frac{B^2}{2} + AB \cos\gamma$$

$$C^2 = A^2 + B^2 - 2AB\cos\gamma$$

Cor: If $\gamma = \pi/2$, $\cos C = \cos A \cos B$



$$P = (\sin A, 0, \cos A)$$

$$Q = (\sin B \cos \gamma, \sin B \sin \gamma, \cos B)$$

$$P \cdot Q = \sin A \sin B \cos \gamma + \cos A \cos B$$

$$= \cos C$$

Law of sines

$$\frac{\sin\alpha}{\sin A} = \frac{\sin\beta}{\sin B} = \frac{\sin\gamma}{\sin C}$$

Sum of angles

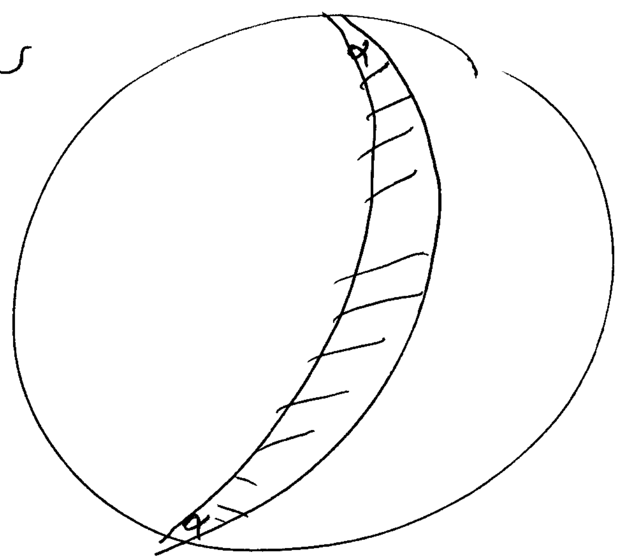
$$\alpha+\beta+\gamma = \pi + \text{Area}(ABC)$$

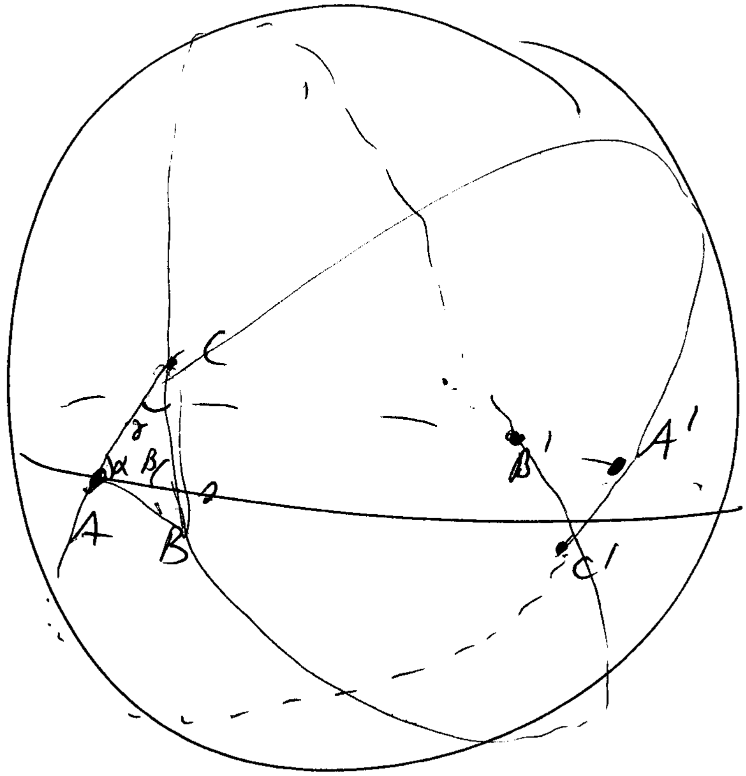
pf Step 1: Lunes

Area of Lune

$$= \frac{\alpha}{2\pi} \cdot 4\pi$$

$$= 2\alpha$$





$$\text{Area}(ABC) + \text{Area}(ABC') = 2\gamma$$

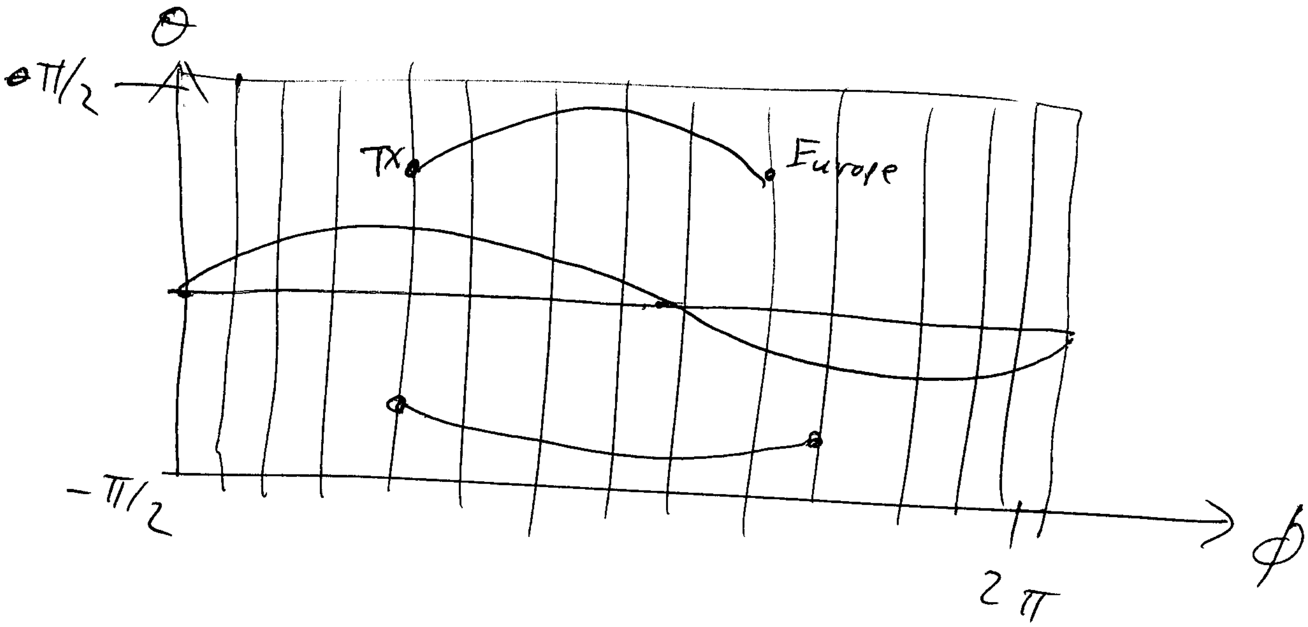
$$\text{Area}(ABC) + \text{Area}(AB'C) = 2\beta.$$

$$\text{Area}(ABC) + \text{Area}(A'BC) = 2\alpha$$

$$\alpha+\beta+\gamma = \frac{3\,\text{Area}(ABC) + \text{Area}(ABC') + \text{Area}(AB'C) + \text{Area}(A'BC)}{2}$$

$$= \frac{2\,\text{Area}(ABC) + \text{Area}(\text{North})}{2}$$

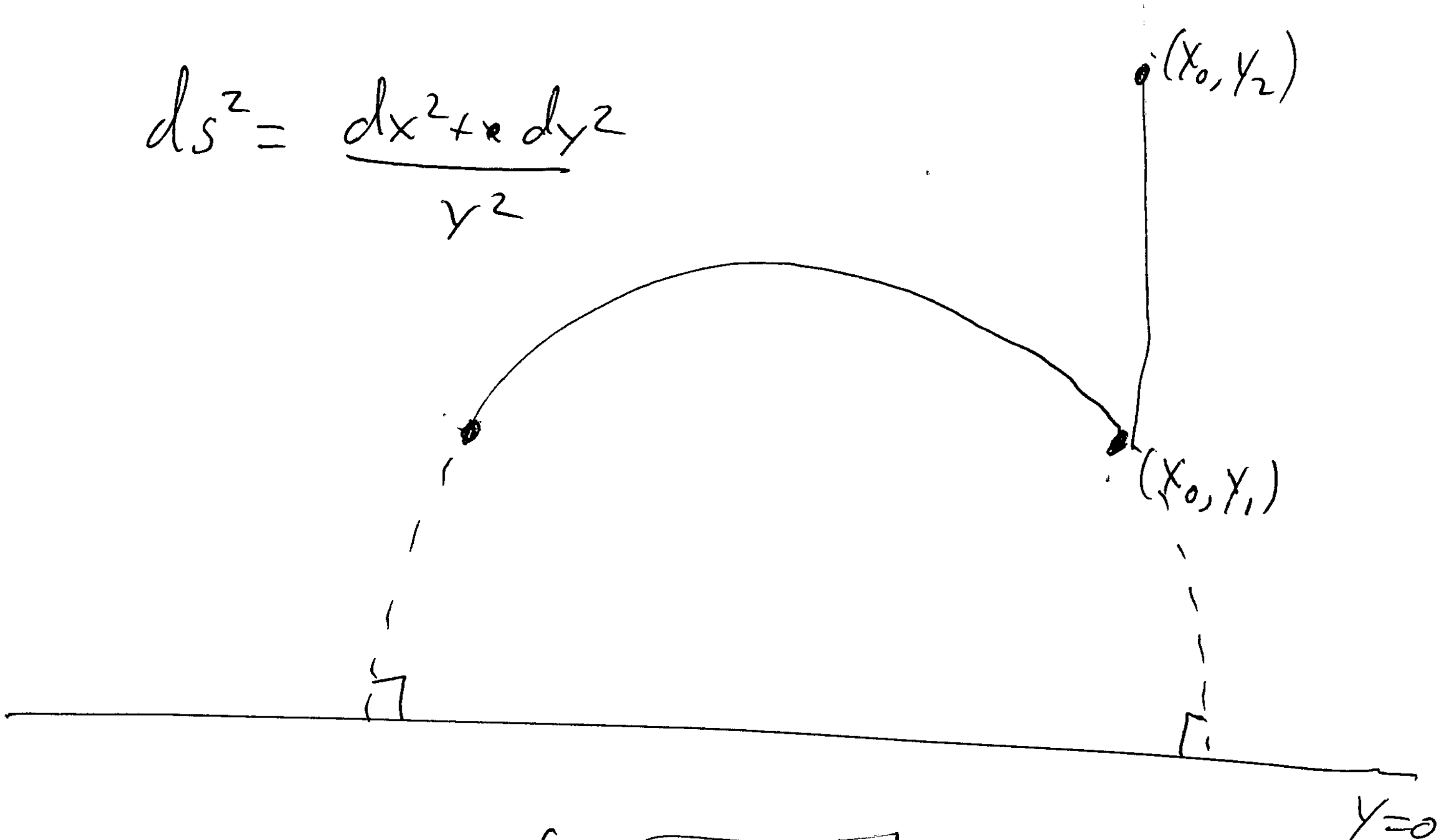
$$= \text{Area}(ABC) + \pi$$


θ
π/2
TX
Europe
−π/2
2π
φ

Upper half plane.

(x,y), $y>0$

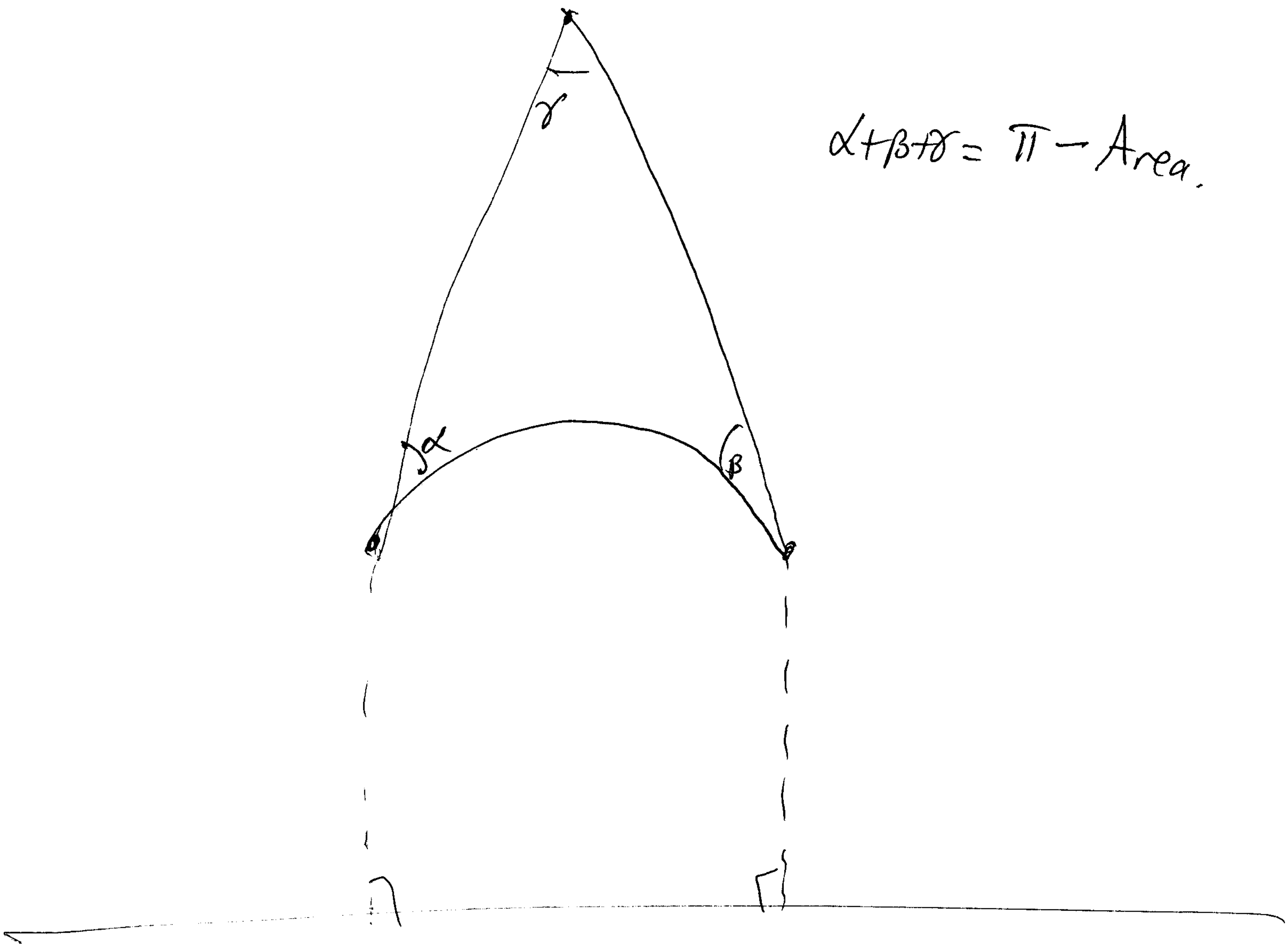
$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

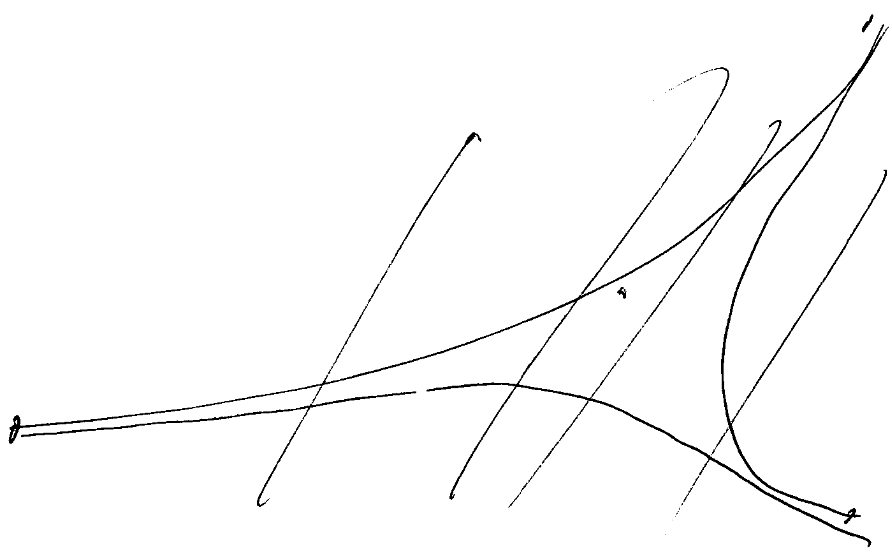


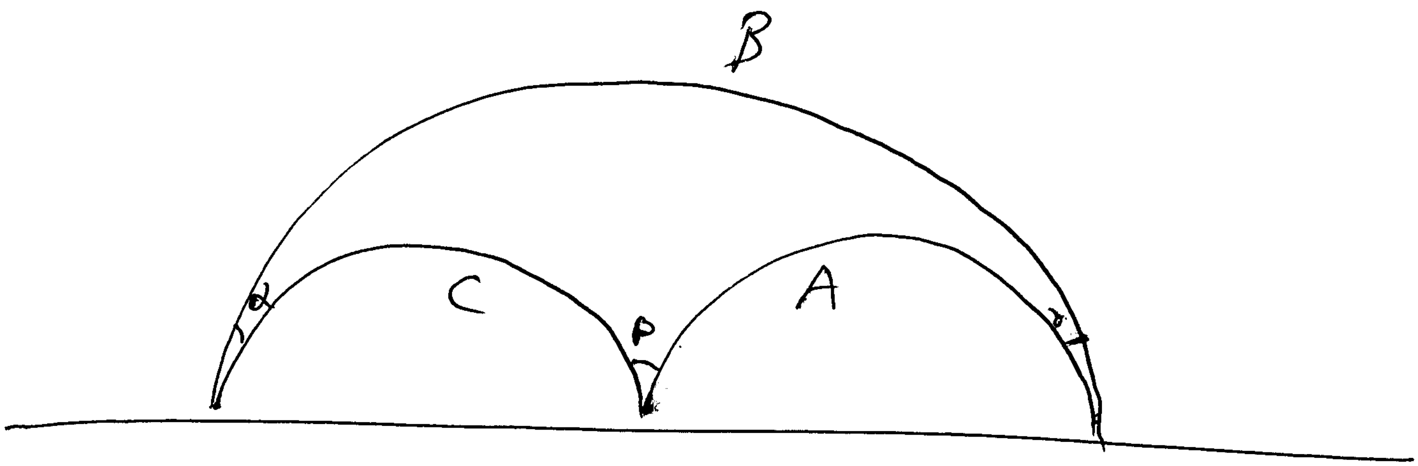
$$\text{distance} = \int \sqrt{\frac{(\dot{x})^2 + (\dot{y})^2}{y^2}} \geq \int \frac{\dot{y}}{y}\,dt$$

$$= \int \frac{dy}{y} = \ln\left(\frac{y_2}{y_1}\right)$$

$$\alpha+\beta+\gamma = \pi - \text{Area}.$$





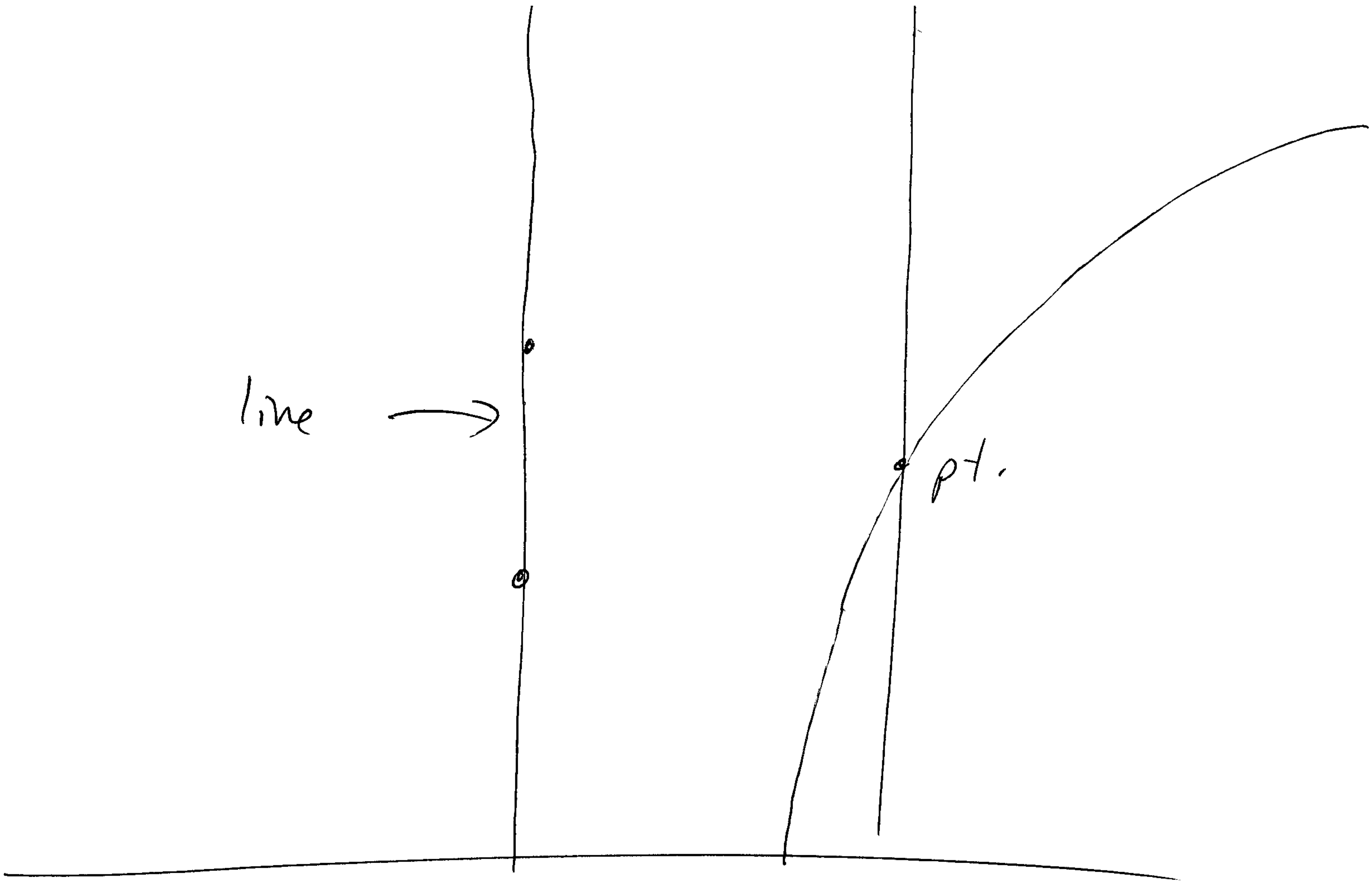


$$\cosh C = \cosh A \cosh B - \sinh A \sinh B \cos\gamma$$

$$1 + \frac{C^2}{2} = \left(1+\frac{A^2}{2}\right)\left(1+\frac{B^2}{2}\right) - AB\cos\gamma$$

$$C^2 = A^2 + B^2 - 2AB\cos\gamma$$

Euclidean limit.


line
pt.