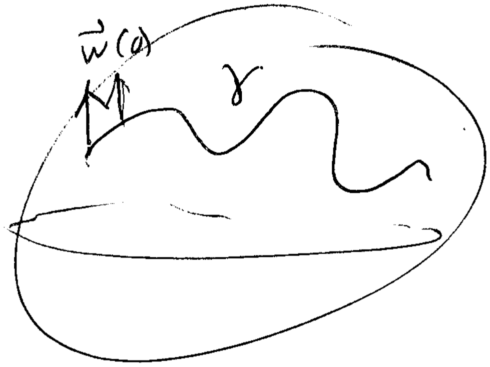
Parallel transport along <u>any</u> curve



Limit of "move a little then project on tangent plane".

$\nabla_\gamma w$ = part of $\frac{d\vec{w}}{dt}$ that is in TS

$x^1 = u$
$x^2 = v$

$$\left(\nabla_\gamma w\right)^i = \frac{dw^i}{dt} + \sum_{jk} \Gamma^i_{jk} w^j \dot{x}^k$$

$$\left(\frac{dw^i}{dt} = -\sum_{jk} \Gamma^i_{jk} w^j \dot{x}^k \right)$$

$$\Gamma^2_{11} = \frac{1}{v} \quad , \quad \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = \frac{-1}{v}$$

$$(0,1) \longrightarrow (1,1)$$

$$\dot{u} = \dot{x}^1 = 1$$

$$\dot{v} = 0 \qquad v = 1$$

$$\frac{dw^1}{dt} = -\cancelto{0}{\Gamma^1_{11} w^1 \dot{x}^1} - \cancel{\Gamma^1_{12} w^1 \dot{x}^2} - \Gamma^1_{21} w^2 \dot{x}^1 - \cancelto{0}{\Gamma^1_{22} w^2 \dot{x}^2}$$

$$= +\frac{1}{v} w^2 = w^2$$

$$\frac{dw^2}{dt} = -\Gamma^2_{11} \dot{x}^1 w^1 - \Gamma^2_{22} \dot{x}^2 w^2 = -\frac{1}{v} w^1 = -w^1$$

$$\frac{d}{dt}\begin{pmatrix} w^1 \\ w^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} w^1 \\ w^2 \end{pmatrix}$$

$$w^1(t) = w^1(0)\cos(t) + w^2(0)\sin(t)$$

$$w^2(t) = -w^1(0)\sin(t) + w^2(0)\cos(t)$$

Vertical $\cancel{}$ $\dot{u}=0$, $u=\text{const.}$

$$\frac{dw^1}{dt} = -\Gamma^1_{12} w^1 \dot{v} = \frac{\dot{v}}{v} w^1$$

$$\frac{dw^2}{dt} = -\Gamma^2_{22} w^2 \dot{v} = \frac{\dot{v}}{v} w^2$$

$$\frac{dw^1}{\cancel{d}w^1} = \frac{dv}{v}$$

$$\ln w^1 = \ln v + c$$

$$w^1 = v \cdot c^1$$

$$w^2 = \cancel{} v \cdot c^2$$

What's a geodesic?

Def 1) $\ddot{\gamma} \perp$ surface

$=$ motion of a particle constrained to surface.

$m\vec{a} = \vec{F} \perp$ surface

$\cancel{m}\vec{a} = \ddot{\gamma} \perp$ surface.

Def 2) Constant-speed curve with $K_g = 0$

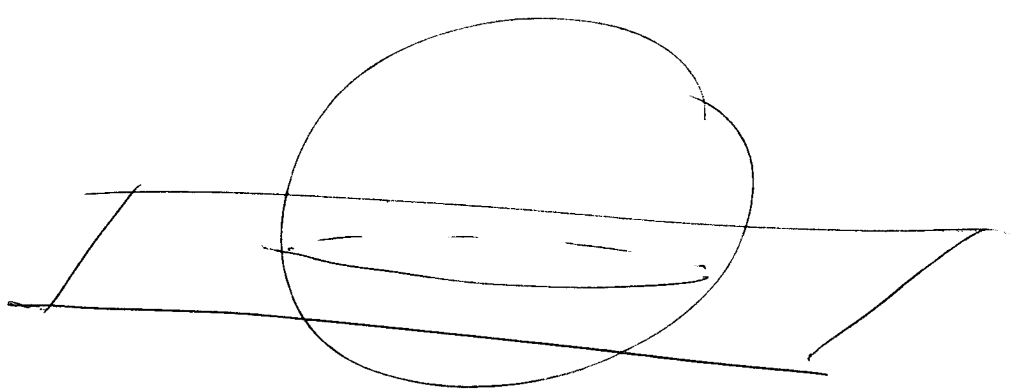
Def 3) Solution to $\nabla_{\dot\gamma} \dot{\gamma} = 0$

$\nabla_{\dot\gamma} \dot{\gamma} =$ tangential part of $\ddot{\gamma}$

Def 4) Constant-speed curve that minimizes* length.

Def 5) Curve that minimizes* $\int_a^b |\dot{\gamma}|^2 \, dt$ ~~ato~~ among paths that take a fixed time to go from $\gamma(a)$ to $\gamma(b)$.

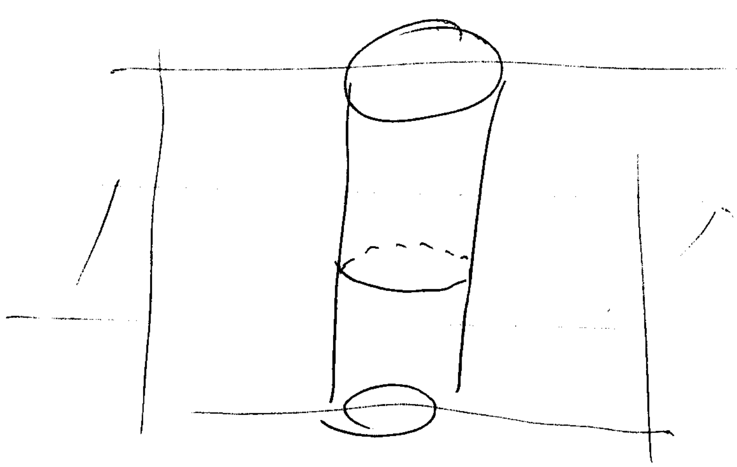
Def A normal section is the curve cut out from a surface by a plane that is normal to surface at each pt. of contact.



Thm normal sections are geodesics

Cor All great circles on S^2 are geodesics.

Ex:



Thm Straight lines are geodesics.

Equations for geodesic.

Form I.

$$\frac{d}{dt}(E\dot{u}+F\dot{v}) = \frac{1}{2}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2)$$

$$= \frac{1}{2}(\dot{u}\ \dot{v})(FFF)_u\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix}$$

$$\frac{d}{dt}(F\dot{u}+G\dot{v}) = \frac{1}{2}(\dot{u}\ \dot{v})(FFF)_v\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix}$$

Form II

$$\ddot{u} = -\sum_{ij}\Gamma_{ij}^1 \dot{x}^i\dot{x}^j$$

$$= -\Gamma_{11}^1(\dot{u})^2 - 2\Gamma_{12}^1(\dot{u})(\dot{v}) - \Gamma_{22}^1(\dot{v})^2$$

$$\ddot{v} = -\Gamma_{11}^2(\dot{u})^2 - 2\Gamma_{12}^2\dot{u}\dot{v} - \Gamma_{22}^2(\dot{v})^2$$

Thm $\exists!$ geodesic with given $\gamma(0)$, $\dot{\gamma}(0)$.

Cor All geodesics on sphere are great circles.

$$\dot{\gamma} = \dot{u}\,\sigma_u + \dot{v}\,\sigma_v$$

$\ddot{\gamma} \perp$ surface.

$$\sigma_u \cdot \ddot{\gamma} = 0$$

$$\sigma_v \cdot \ddot{\gamma} = 0$$

$$\sigma_u \cdot \ddot{\gamma} = \frac{d}{dt}(\sigma_u \cdot \dot{\gamma}) - \frac{d\sigma_u}{dt} \cdot \dot{\gamma}$$

$$= \frac{d}{dt}(\dot{u}E + \dot{v}F) - \frac{d\sigma_u}{dt} \cdot \dot{\gamma}$$

$$= \frac{d}{dt}(\dot{u}E + \dot{v}F) - \underbrace{(\sigma_{uu}\dot{u} + \sigma_{uv}\dot{v}) \cdot (\dot{u}\sigma_u + \dot{v}\sigma_v)}_{J'}$$

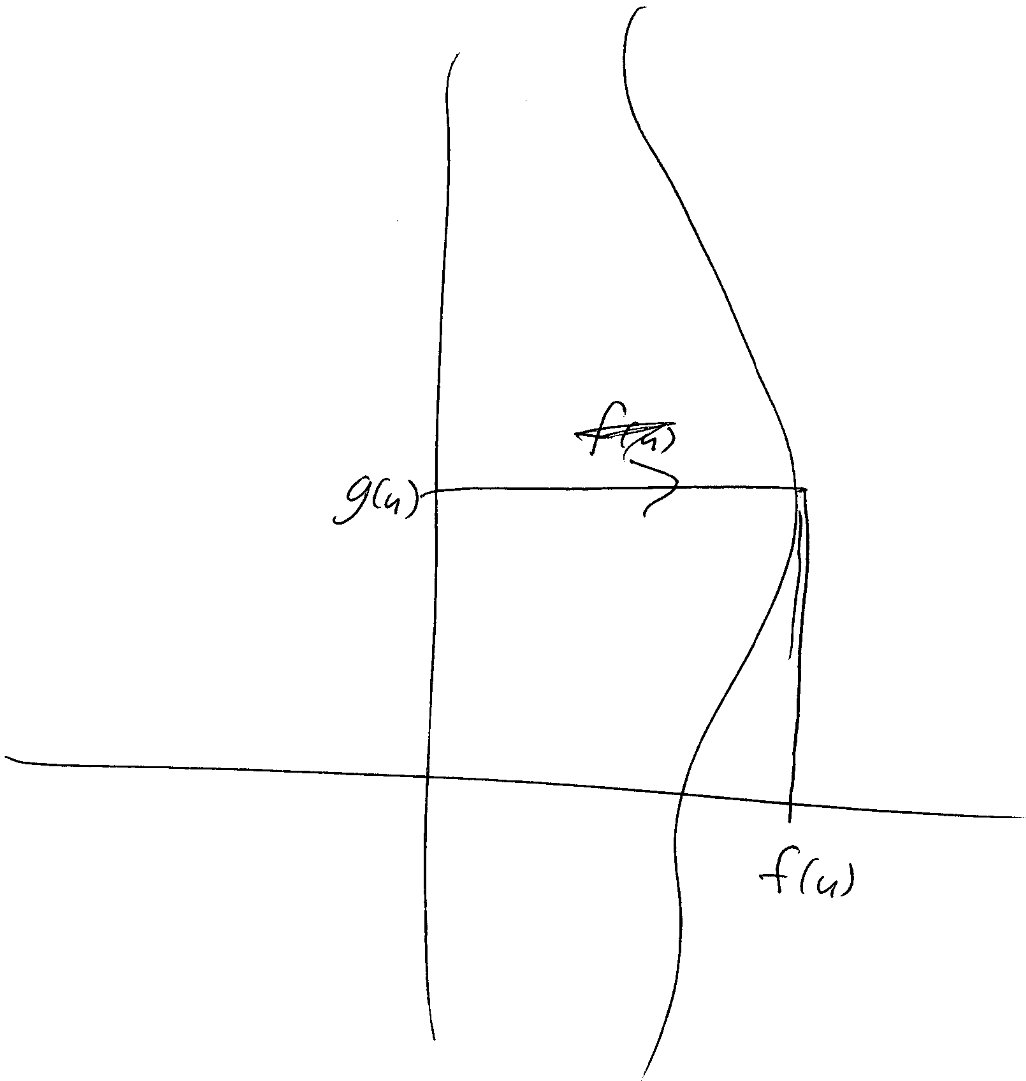
$$J' = (\sigma_{uu} \cdot \sigma_u)\,\dot{u}^2 + (\sigma_{uu} \cdot \sigma_v + \sigma_{uv} \cdot \sigma_u)\,\dot{u}\dot{v} + (\sigma_{uv} \cdot \sigma_v)\,\dot{v}^2$$

$$= \frac{1}{2}(\sigma_u \cdot \sigma_u)_u\,\dot{u}^2 + (\sigma_u \cdot \sigma_v)_u\,\dot{u}\dot{v} + \frac{1}{2}(\sigma_v \cdot \sigma_v)_u\,(\dot{v})^2$$

$$= \frac{1}{2}\left(E_u \dot{u}^2 + 2F_u \dot{u}\dot{v} + G_u \dot{v}^2\right)$$

Same argument for $\frac{d}{dt}(F\dot{u} + G\dot{v})$

Thm ~~Geodes~~ Local Isometries map geodesics to geodesics.



Surface of revolution.

$$\sigma(u,v) = (f(u)\cos v,\ f(u)\sin(v),\ g(u))$$

$f_u^2 + g_u^2 = 1$ $\qquad$ $ds^2 = 1 + f^2 dv^2$

$E = 1,\ F = 0,\ G = f^2(u)$

$$\dot{(E\dot{u}+F\dot{v})} = \ddot{u} = \frac{1}{2} G_u(\dot{v})^2 = f f_u(\dot{v})^2$$

$$\dot{(F\dot{u}+G\dot{v})} = \dot{(f^2\dot{v})} = 0$$

$$f^2\dot{v} = \text{const.}$$

Conservation of angular momentum.