Geodesic Equations

$$\frac{d}{dt}\left(E\dot u + F\dot v\right) = \frac{1}{2}\begin{pmatrix}\dot u & \dot v\end{pmatrix}\begin{pmatrix}E_u & F_u \\ F_u & G_u\end{pmatrix}\begin{pmatrix}\dot u \\ \dot v\end{pmatrix}$$

$$\frac{d}{dt}\left(F\dot u + G\dot v\right) = \frac{1}{2}\begin{pmatrix}\dot u & \dot v\end{pmatrix}\begin{pmatrix}E_v & F_v \\ F_v & G_v\end{pmatrix}\begin{pmatrix}\dot u \\ \dot v\end{pmatrix}$$

$$\ddot u + \sum_{jk} \Gamma^1_{jk}\, \dot x^j \dot x^k = 0$$

$$\ddot v + \sum_{jk} \Gamma^2_{jk}\, \dot x^j \dot x^k = 0$$

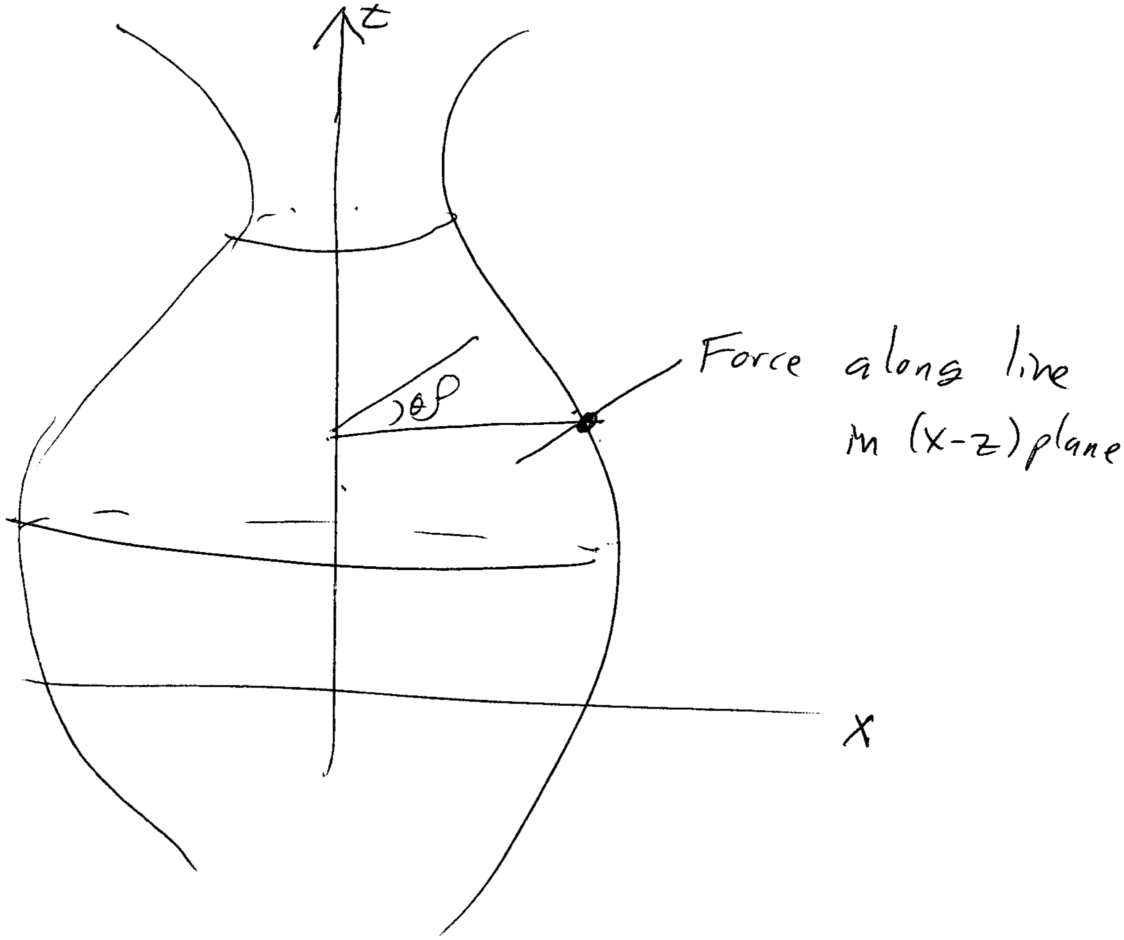
Surface of revolution

$$E = 1, \quad F = 0, \quad G = f^2$$

$$\frac{d}{dt}\left(\dot u\right) = \frac{1}{2}\, \dot v^2 \left(2 f f_u\right) = f f_u \dot v^2$$

$$\frac{d}{dt}\left(f^2 \dot v\right) = 0 \qquad f^2 \dot v = \text{constant} = \Omega$$

$= \text{angular momentum.}$



$$\vec{\tau} = \text{torque} = \vec{X} \times \vec{F} = (\sim, 0, \sim) \times (\sim, 0, \sim)$$

$$= (\sim, \sim, 0)$$

No torque in z-direction, so angular momentum in z direction is conserved.

$$m(x\dot{y} - y\dot{x}) = m\rho^2\dot{\theta} = \underline{m\rho^2\dot{v}} \qquad \begin{pmatrix} \theta = v \\ \rho = f(u) \end{pmatrix}$$

Suppose $\langle \dot\gamma, \dot\gamma \rangle = 1$

$$\dot u^2 + f^2 \dot v^2 = 1$$

$$f^2 \dot v = \Omega$$

$$\dot v = \frac{\Omega}{f^2}$$

$$\dot u^2 + \frac{\Omega^2}{f^2} = 1$$

$$\dot u^2 = 1 - \frac{\Omega^2}{f^2}$$

$\cancel{|f| < \Omega}$ $\boxed{f \geq |\Omega|}$

$$\dot u = \pm\sqrt{1 - \frac{\Omega^2}{f^2}}$$

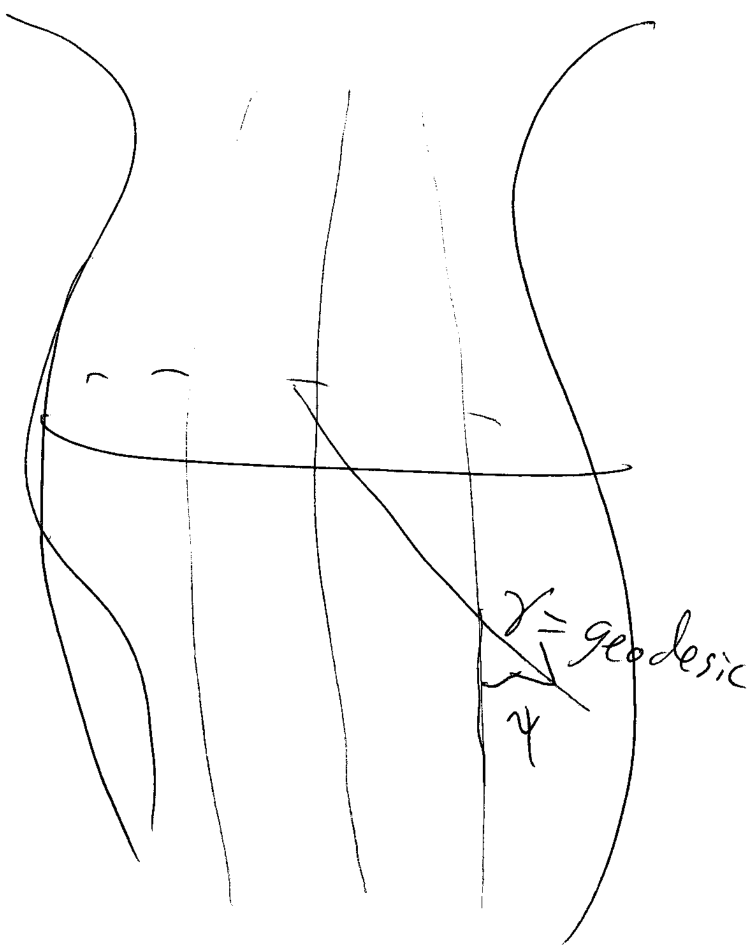
$$\underbrace{\frac{1}{2}\dot u^2}_{\text{Kinetic}} + \underbrace{\frac{1}{2}\frac{\Omega^2}{f^2}}_{\text{Potential}} = \underbrace{\frac{1}{2}}_{\text{total energy}}$$

$$\ddot u = \text{“force”} = -\frac{d}{du}\left(\frac{1}{2}\frac{\Omega^2}{f^2}\right) = \frac{\Omega^2 f_u}{f^3} = \dot v^2 f f_u$$

On sphere, $f = \cos(u)$

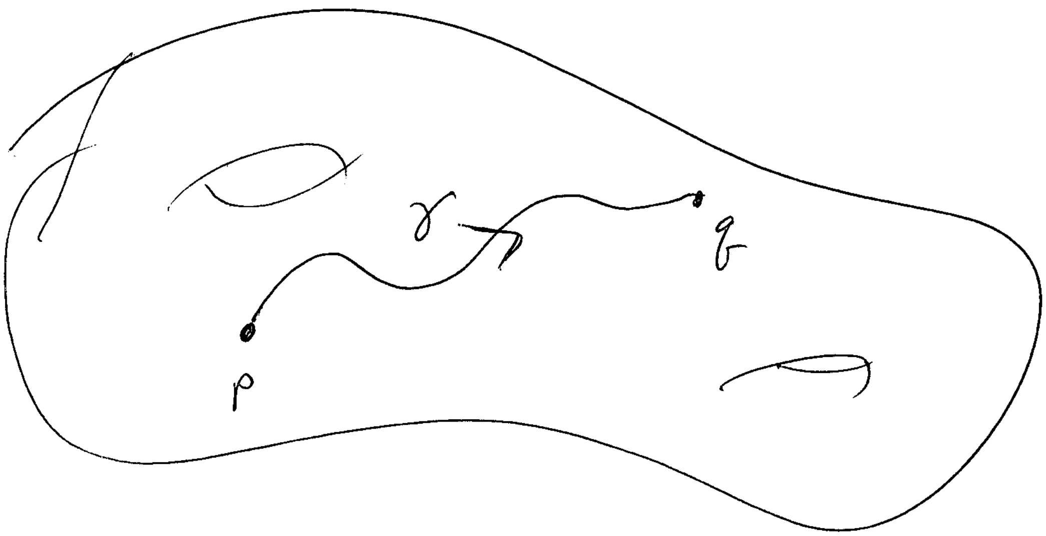
$g = \sin(u)$.

$$\ddot{u} = \frac{\Omega^2(-\sin u)}{\cos^3 u}$$



Clairaut's Thm: $\rho \sin\psi =$ constant.

pf: $\rho \sin\psi = \Omega$.



$\gamma(0)=p, \quad \gamma(T)=q.$

$$\mathcal{S} = \int_0^T \frac{1}{2}\langle \dot\gamma, \dot\gamma \rangle \, dt =$$

Energy of the path (math)

OR

Action of the path.

Problem: find γ that minimizes $\mathcal{S}$

Lemma: If length of γ is L,

$$\mathcal{A} \geq \frac{L^2}{2T}$$, with equality iff

constant speed.

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pf 1: $$\mathcal{A} = \frac{1}{2}\int_0^T \left(\frac{ds}{dt}\right)^2 dt$$

$$= \frac{1}{2}\int_0^T \left(\dot{s} - \frac{L}{T} + \frac{L}{T}\right)^2 dt$$

$$= \frac{1}{2}\int_0^T \left(\dot{s} - \frac{L}{T}\right)^2 + \frac{L^2}{T^2} + 2\frac{L}{T}\left(\dot{s} - \frac{L}{T}\right) dt$$

$$= \frac{1}{2}\int_0^T \left(\dot{s} - \frac{L}{T}\right)^2 + \frac{L^2}{T^2}\, dt$$

$$\geq \frac{1}{2}\int_0^T \frac{L^2}{T^2}\, dt = \frac{L^2}{2T}$$

$$\$ = \frac{1}{2} \int_0^T (\dot{s})^2 \, dt$$

$$s(t) \rightarrow s(t) + \delta s(t)$$

$$\delta(\dot{s}) = \dot{(\delta s)}$$

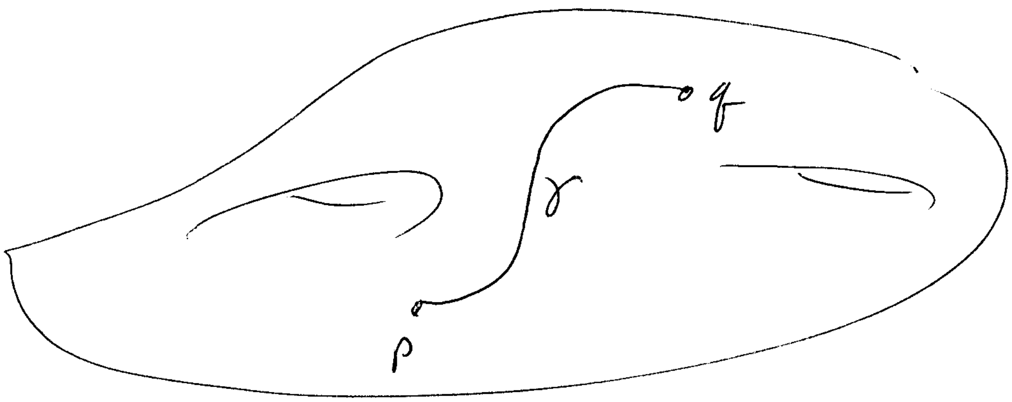
$$\delta \$ = \int_0^T \dot{s} \, \delta \dot{s} \, dt$$

$$= \dot{s} \, \delta s \Big|_0^T - \int_0^T \ddot{s} \, (\delta s) \, dt$$

$$= \int_0^T \ddot{s} \, \delta s \, dt = 0$$

for all possible δs.

So $\ddot{s} = 0$, constant speed.



$u(t)$

$v(t)$

$$u \to u + \delta u \qquad \dot{u} \to \dot{u} + (\dot{\delta u})$$

$$v \to v + \delta v \qquad \dot{v} \to \dot{v} + (\dot{\delta v})$$

$$\$ \to \$ + \delta \$ \qquad \text{Set } \delta \$ = 0.$$

$$\$ = \frac{1}{2}\int_0^T \left(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2\right) dt$$

term 1

$$\delta \$ = \frac{1}{2}\int_0^T (\delta E)\dot{u}^2 + 2\delta F(\dot{u}\dot{v}) + \delta G(\dot{v})^2 \, dt$$

$$+ \int_0^T \left(E\dot{u}\,\delta\dot{u} + \cancel{2}F\dot{u}\,\delta\dot{v} + F\dot{v}\,\delta\dot{u} + G\dot{v}\,\delta\dot{v}\right) dt$$

term 2.

Term 1: $\delta E = E_u \, \delta u + E_v \, \delta v$

$$\text{Term 1} = \frac{1}{2} \int_0^T \Big[\left(E_u \dot{u}^2 + 2F_u \dot{u}\dot{v} + G_u \dot{v}^2 \right) \delta u + \left(E_v \dot{u}^2 + 2F_v \dot{u}\dot{v} + G_v \dot{v}^2 \right) \delta v \Big] \, dt$$

$$\text{Term 2} = \int_0^T (E\dot{u} + F\dot{v}) \, \delta \dot{u} + (F\dot{u} + G\dot{v}) \, \delta \dot{v} \; dt$$

$$= (E\dot{u} + F\dot{v}) \, \delta u + (F\dot{u} + G\dot{v}) \, \delta v \Big|_0^T$$

$$+ \int \Big[- \left(\frac{d}{dt} (E\dot{u} + F\dot{v}) \right) \delta u - \left(\frac{d}{dt} (F\dot{u} + G\dot{v}) \right) \delta v \Big] \, dt$$

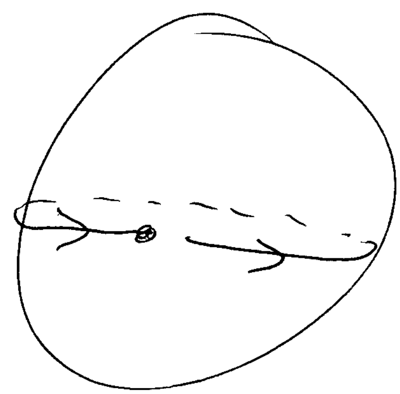
$$\delta \$ = \int \Big[\left(\frac{1}{2} \left[(E_u \dot{u}^2 + 2F_u \dot{u}\dot{v} + G_u \dot{v}^2 \right] - \frac{d}{dt} (E\dot{u} + F\dot{v}) \right) \delta u + \left(\quad \text{similar} \quad \right) \delta v \Big] \, dt$$

$\delta \$ = 0 \iff$ Geodesic equations.

Geodesics are stationary pts of cost function.

Length-minimizers are geodesics.

But geodesics don't always minimize length.



Functional $\mathcal{L}(q_1, q_2, \ldots q_n, \dot{q}_1, \ldots \dot{q}_n, t)$

For geodesic, $q_1 = u(t)$

$q_2 = v(t)$

$$\mathcal{L} = \frac{1}{2} E\dot{u}^2 + F\dot{u}\dot{v} + \frac{1}{2} G\dot{v}^2$$

$$S = \text{Action} = \int_{t_0}^{t_1} \mathcal{L}(\vec{q}, \dot{\vec{q}}, t)\, dt$$

Goal: Mimimize action.

$$\delta S = \int \delta \mathcal{L}\, dt = \sum \int \left(\frac{\partial \mathcal{L}}{\partial q_i} \delta q_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt$$

$$= \sum_i \int \left[\left(\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) \delta q \right] dt.$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i}$$ Euler-Lagrange equs.

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = p_i = \text{momentum.}$$

Ex: $\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$$p_x = \frac{\delta \mathcal{L}}{\delta \dot{x}} = m\dot{x} \qquad p_y = m\dot{y}, \quad p_z = m\dot{z}$$

$$\frac{d}{dt} p_i = \frac{\partial \mathcal{L}}{\partial q_i}$$

$\boxed{\text{If } \mathcal{L} \text{ is independent of } q_i,\ p_i \text{ is conserved.}}$

Geodesics

$$\mathcal{L} = \frac{1}{2}E\dot{u}^2 + F\dot{u}\dot{v} + \frac{1}{2}G\dot{v}^2$$

$$p_1 = E\dot{u} + F\dot{v}$$

$$p_2 = F\dot{u} + G\dot{v}$$

$$\frac{dp_1}{dt} = \frac{\partial \mathcal{L}}{\partial u} \qquad \frac{dp_2}{dt} = \frac{\partial \mathcal{L}}{\partial v}$$