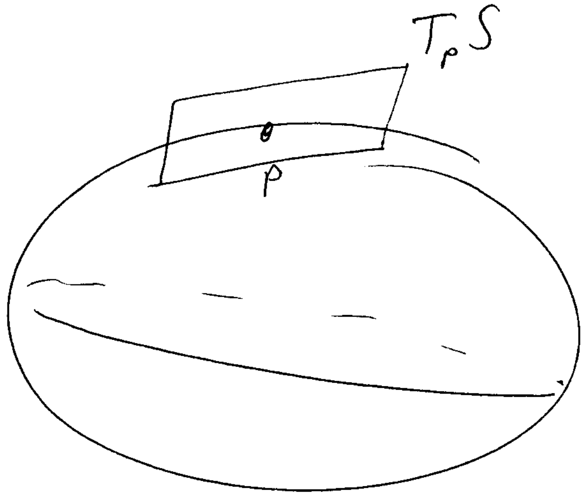


$$\widetilde{\exp}_p : T_pS \times \mathbb{R} \to S.$$
$$\vec{v}, t$$

Let γ be geodesic with $\gamma(0)=p$, $\dot{\gamma}(0)=\vec{v}$

Let $\widetilde{\exp}_p(\vec{v},t) = \gamma(t)$.

$$\widetilde{\exp}_p(s\vec{v},t) = \widetilde{\exp}_p(\vec{v},st)$$

$\tilde{\gamma}(t) = \gamma(st)$ is geodesic w/ initial velocity $s\vec{v}$.

$$\widetilde{\exp}_p(\vec{v},t) = \widetilde{\exp}_p(t\vec{v},1)$$

Def $\exp_p(\vec{v}) = \widetilde{\exp}_p(\vec{v},1)$

$\exp_p : T_p S \longrightarrow S$

$D \exp_p \big|_0 : T_0(T_p S) \to T_p S$

$D\exp_p \big|_0 : T_p(S) \to T_p(S)$ is identity.

$\exp_p$ is local homemorphism between nbhd of 0 in $T_p S$ & nbhd of p in S.

Pick orthonormal basis $\vec{e}_1, \vec{e}_2$ of $T_p S$.

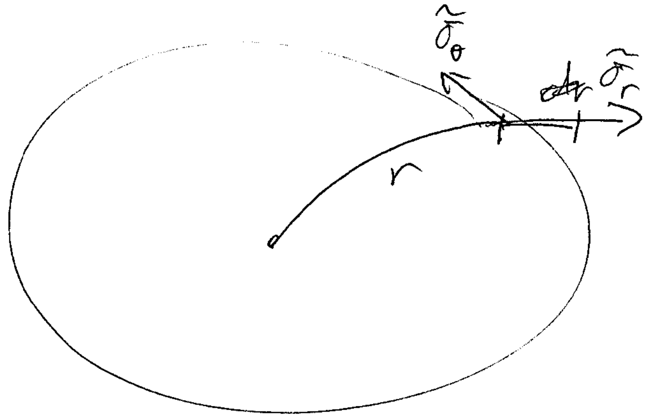
Define $\sigma(u,v) = \exp_p(u\vec{e}_1 + v\vec{e}_2)$

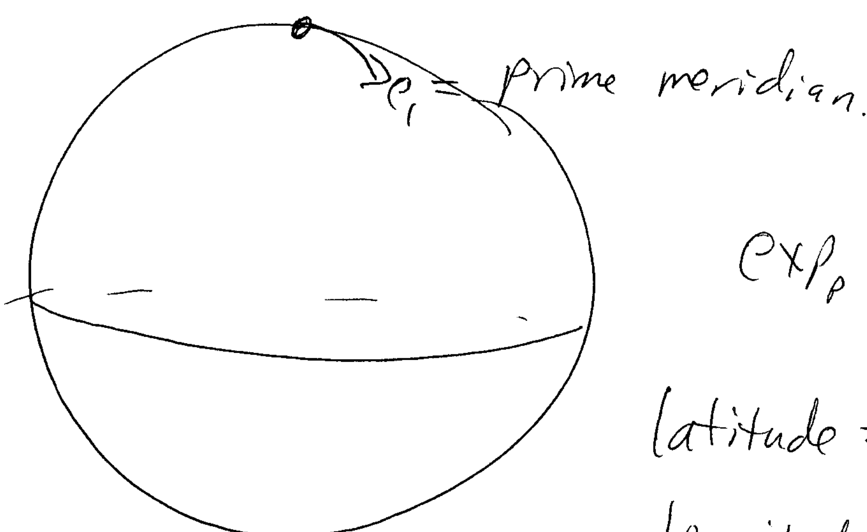
$u = r\cos\theta$

$v = r\sin\theta$.

$$\hat{\sigma}(r,\theta) = \sigma(r\cos\theta, r\sin\theta)$$
$$= \exp_p\left(r(\cos\theta\,\vec{e}_1 + \sin\theta\,\vec{e}_2)\right)$$

Prop $\tilde{E} = 1$, $\tilde{F} = 0$. $\tilde{G}$ = function of r, θ.



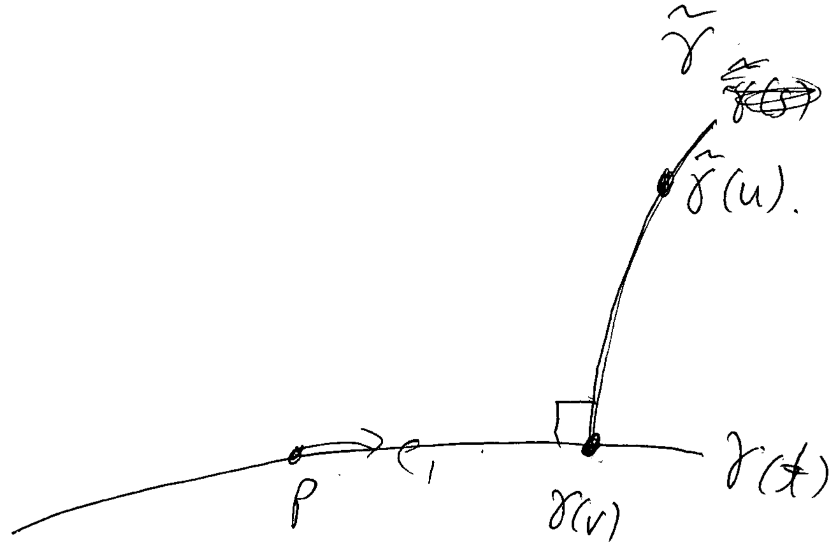


$\exp_p(r, \theta)$

latitude $= \frac{\pi}{2} - r$

longitude $= \theta$.

$$\begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 r \end{pmatrix}$$



$\sigma(u,v)$ = result of following γ for time v,
turning $\perp$, follow geodesic $\tilde\gamma$ for time u.

Prop well-defined. ($\sigma_u(0,0)=\dot\gamma(0)$, $\sigma_v(0,0)\perp\dot\gamma(0)$)

Prop $E=1, F=0$.

Pf. i) $\sigma_u(u,v)=\dot{\tilde\gamma}(u)$ has length 1, so $E=1$.

ii) $\sigma_v(0,v)\perp\sigma_u(0,v)$, so $F(0,v)=0$

iii) $\tilde\gamma$ is a geodesic, so

$$\frac{d}{dt}(F\dot u+G\dot v)=\frac{1}{2}\partial_v(\underbrace{E}_{1}\dot u^2+2F\dot u\dot v+G\dot v^2)$$

But $\dot v=0$, $\dot u=1$, so

$$\frac{d}{dt}F=\frac{1}{2}\partial_u(1)=0$$

$F_u=0$, so $F=0$ everywhere.

If $S = S^2$, $\rho = \rho^+$ on equator, γ = equator,

V = ~~ta~~ longitude, u = latitude.

Notions of curvature

1) Extent to which ∇_u, ∇_v fail to commute.

$$[\nabla_u, \nabla_v] = C_1 \overset{\sqrt{EG-F^2}}{\cancel{\sqrt{EF-G^2}}} \cdot (\text{Rotation CW by } 90^\circ)$$

2) Rotation/unit area induced by parallel transport.

Effect of parallel transport = rotation CCW by $\iint_W C_2 \, d(\text{Area})$

(Local) Gauss-Bonnet thm

3) $K = \dfrac{LN-M^2}{EG-F^2}$

4) Product of principal curvatures

5) det (dGauss map) = det W

6) Ugly formula involving E, F, G & derivatives of E, F, G.
(Theorema Egregium)

6a) When $F = 0$

$$-\frac{1}{2\sqrt{EG}}\left(\left(\frac{G_u}{\sqrt{EG}}\right)_u + \left(\frac{E_v}{\sqrt{EG}}\right)_v\right)$$

6b) When $E=1$ and $F=0$

$$-\frac{1}{\sqrt{G}}\left(\frac{\partial^2 \sqrt{G}}{\partial u^2}\right)$$

6c) Surface of revolution. $E=1, F=0, G=f^2$

$$-\frac{f_{uu}}{f}$$

7) Quantized (topological) geometric quantity

$$\iint_S K \, d(\text{Area}) = 2\pi \left(\begin{matrix}\text{Euler}\\ \text{characteristic}\end{matrix}\right)$$

(global) Gauss-Bonnet

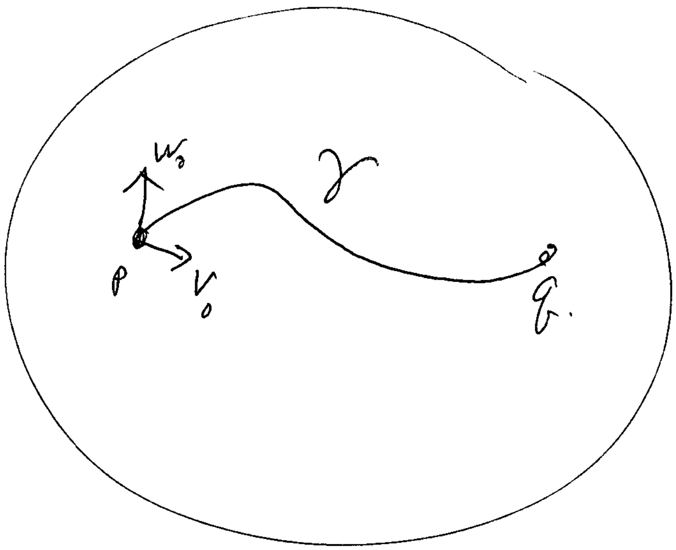
Game plan:

Study 2. Then Show $C_1 = C_2$.

Then find formula for C_1

Then show $C_1 = K$ (HW).

Then get 7 from 2.



Π_γ = parallel transport from p to q.

<u>Prop.</u> Π_γ preserves length & angles.

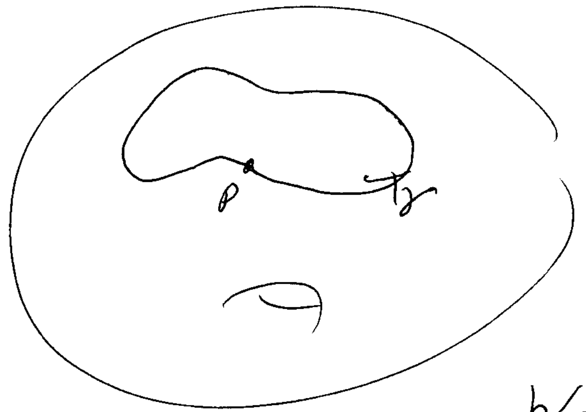
$$\frac{d}{dt}\langle v, w\rangle = \langle \dot{v}, w\rangle + \langle v, \dot{w}\rangle$$

But $\dot{v} = \nabla_\gamma v + (\text{normal to } S)$

$$\langle \dot{v}, w\rangle = \langle \nabla_\gamma v, w\rangle + \cancelto{0}{\langle \text{normal to } S, w\rangle}$$

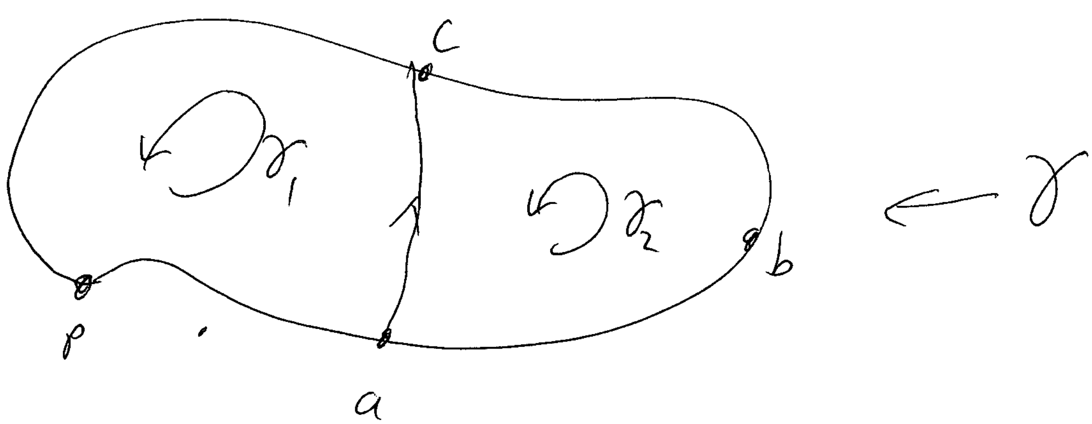
$$\frac{d}{dt}\langle v, w\rangle = \langle \nabla_\gamma v, w\rangle + \langle v, \nabla_\gamma w\rangle$$

$$= 0 + 0 = 0.$$



Π_γ is a rotation b/c it preserves lengths & angles.

Π_γ commutes w/ rotations.



Claim: angle of rotation by γ = rotation by γ_1 + rotation by γ_2