Notions of curvature

1) $[\nabla_1, \nabla_2] = C_1 \cdot \sqrt{EG-F^2} \cdot \cancel{Ro}$ (CW rotation by $90°$)

$= -C_1 \sqrt{EG-F^2} \cdot$ (CCW rotation by $90°$)

Good: a) Gives formula for C_1 in terms of Γ_{ij}^k, $\partial_\ell \Gamma_{ij}^k$

b) Invariant under local isometries.

Bad: Formulas are ugly. — impedes intuition.

Change-of-coordinates?

2) C_2 = angle of CCW rotation / unit area from parallel transport around loop.

Good: Invariant under everything.

Bad: no formula

3) $K = \det W = (LN-M^2)/(EG-F^2) = K_1 K_2$

Good: Coordinate-independent, can calculate

Bad: Extrinsic.

Game Plan

4) HW: $C_1 = K$

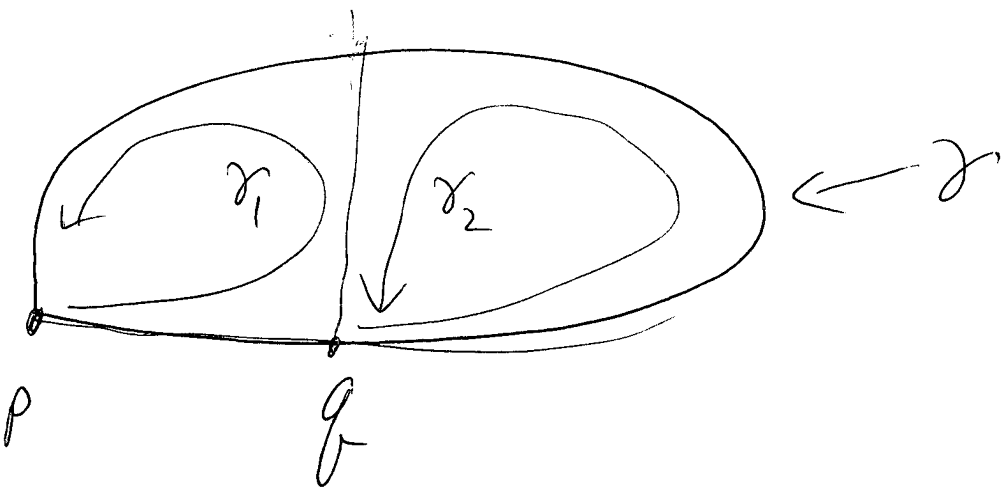
1) Show C_2 well-defined.
2) Show $C_1 = C_2$
3) Get formulas for C_1

<u>Thm</u> Parallel transport preserves length & angle.

<u>Cor 1</u> On oriented manifold, or on small loop, Parallel transport around closed loop is rotation.

<u>Cor 2</u> Π around different loops commute. and Π along path commutes w/ rotation.

Rotate at p then apply Π_{pq} = Π_{pq} and then rotate at q.



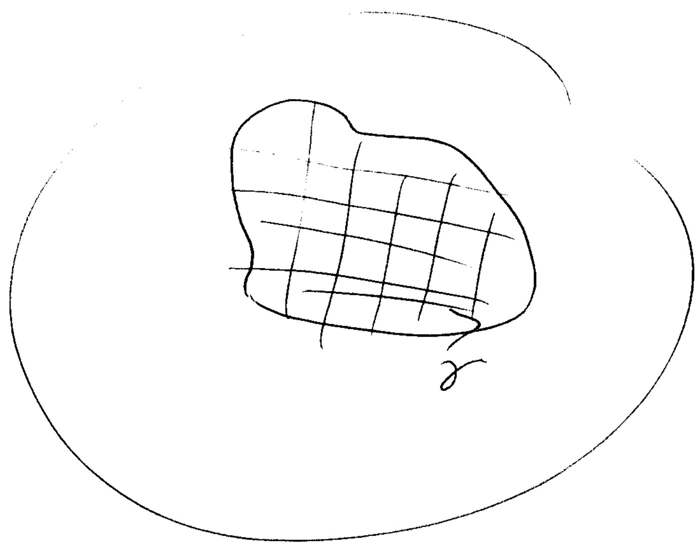
Claim: ~~Ro~~ Angle of rotation from $\underbrace{\Pi_\gamma}_{\theta}$

$= \text{angle from } \underbrace{\Pi_{\gamma_1}}_{\theta_1} + \text{angle from } \underbrace{\Pi_{\gamma_2}}_{\theta_2}$

Pf: $\Pi_\gamma = \Pi_{\gamma_1} \Pi_{qp} \Pi_{\gamma_2} \Pi_{pq}$

$= (\text{rotate by } \theta_1)\, \Pi_{qp}\, (\text{rotate by } \theta_2)\, \Pi_{pq}$

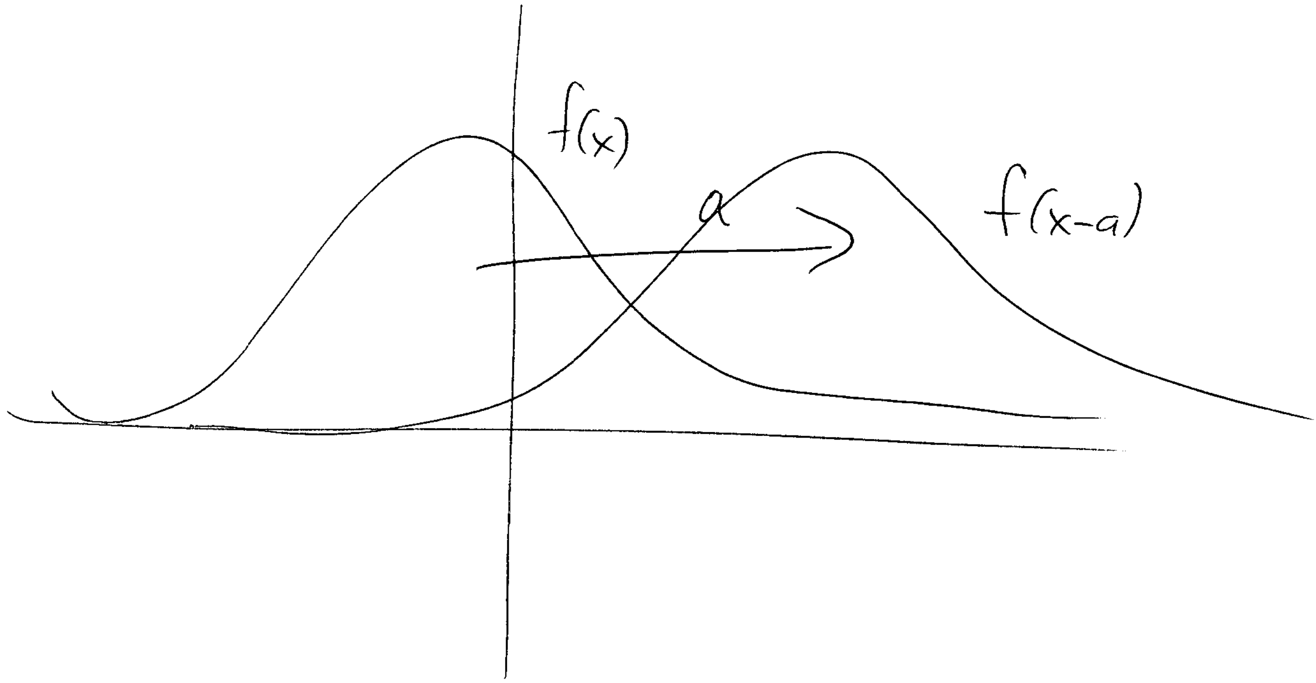
$= \text{rotate by} (\theta_1 + \theta_2)$



$$\text{Angle from } \Pi_\gamma = \sum_{\text{pieces}} \left(\begin{array}{c}\text{angle from each} \\ \text{piece}\end{array}\right)$$

$$= \iint \left(\frac{\text{rotation}}{\text{unit area}}\right) d\,\text{Area}$$

C_2

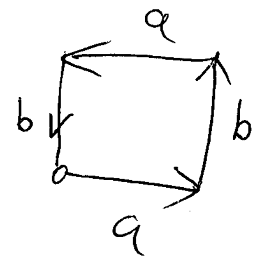


$$f(x-a) = \sum_{n=0}^{\infty} \cancel{f(x} \, f^{n}(x) \, \frac{(-a)^n}{n!}$$

$$= \left[\sum_{n=0}^{\infty} \left(\frac{(-a)^n}{n!} \frac{d^n}{dx^n} \right) \right] f(x)$$

$$= \boxed{e^{-a\frac{d}{dx}}} \; f(x)$$

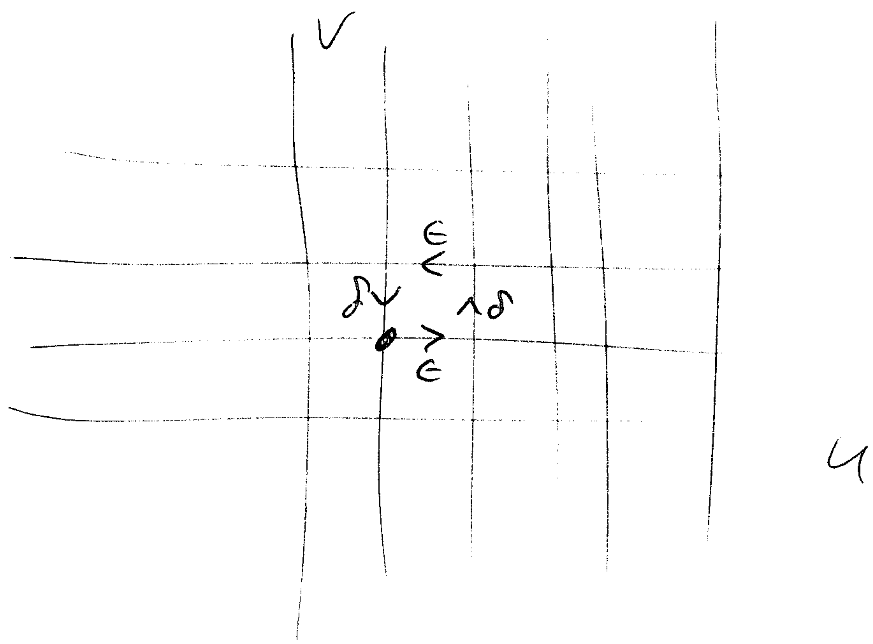
$f(x,y)$



$$e^{b\frac{d}{dy}}\, e^{a\frac{d}{dx}}\, e^{-b\frac{d}{dy}}\, e^{-a\frac{d}{dx}} = 1$$

because $\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] = 0.$

Surface. $\vec{W}(u,v) = w^1\sigma_u + w^2\sigma_v$



$$\Pi_{\substack{u\to u+a\\ v=\text{const}}} = e^{-a\nabla_1}$$

$$\Pi_{\substack{u=\text{const}\\ v\Rightarrow v+b}} = e^{-b\nabla_2}$$

$$e^{\delta\nabla_2}\, e^{\epsilon\nabla_1}\, e^{-\delta\nabla_2}\, e^{-\epsilon\nabla_1}$$

$$= (1+\delta\nabla_2)(1+\epsilon\nabla_1)(1-\delta\nabla_2)(1-\epsilon\nabla_1) + \text{higher order}$$

$$= 1 + \epsilon\delta(\nabla_2\nabla_1 - \nabla_2\nabla_1 - \nabla_1\nabla_2 + \nabla_2\nabla_1) + \text{h.o.}$$

$$= 1 + \epsilon\delta(\nabla_2\nabla_1 - \nabla_1\nabla_2) = 1 - \epsilon\delta[\nabla_1, \nabla_2]$$

Rotation by small angle $\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + O(\theta^2)$$

$$1 - \underset{+\,h.o.}{\epsilon\delta[\nabla_1, \nabla_2]} = 1 + \theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + h.o.$$

$$[\nabla_1, \nabla_2] = -\left(\lim_{\epsilon,\delta \to 0} \frac{\theta}{\epsilon\delta}\right) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$d\,\text{Area} = du\,dv\,\sqrt{EG-F^2}$$

in orthonormal basis.

$$[\nabla_1, \nabla_2] = -\sqrt{EG-F^2}\,\left(\frac{\theta}{\text{area}}\right) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

C_2

$$\boxed{[\nabla_1, \nabla_2] = -\sqrt{EG-F^2}\; C_2 \cdot \text{rotation CCW by } 90^\circ}$$

$$\Longrightarrow C_1 = C_2$$

Rotation by 90° in σ_u, σ_v basis is

$$\cancel{\frac{1}{\sqrt{EG-F^2}}\begin{pmatrix} -F & -G \\ E & F \end{pmatrix}} = \frac{1}{\sqrt{EG-F^2}}\begin{pmatrix} -F & -G \\ E & F \end{pmatrix}$$

$$\beta = \frac{(-F\sigma_u + E\sigma_v)}{\sqrt{EG-F^2}}$$

$$\beta \cdot \sigma_u = \frac{-FE + EF}{\sqrt{EG-F^2}}$$

$$\beta \cdot \beta = \frac{F^2E + E^2G - 2FEF}{EG-F^2}$$

$$= \frac{E^2G - EF^2}{EG-F^2} = E = \sigma_u \cdot \sigma_u$$

$$\alpha = \frac{-G\sigma_u + F\sigma_v}{\sqrt{EG-F^2}}$$

$$\alpha \cdot \sigma_v = -\frac{GF + FG}{\sqrt{\quad}} = 0$$

$$\alpha \cdot \alpha = \frac{G^2E + F^2G - 2FGF}{EG-F^2} = G$$

$$(\nabla_1 w) = \begin{pmatrix} (\nabla_1 w)^1 \\ (\nabla_1 w)^2 \end{pmatrix} = \frac{\partial}{\partial u}\begin{pmatrix} w^1 \\ w^2 \end{pmatrix} + \begin{pmatrix} \Gamma^1_{11} & \Gamma^1_{12} \\ \Gamma^2_{11} & \Gamma^2_{12} \end{pmatrix}\begin{pmatrix} w^1 \\ w^2 \end{pmatrix}$$

$$\nabla_1 = \partial_1 + A_1$$

$$\nabla_2 = \partial_2 + A_2$$

$$A_2 = \begin{pmatrix} \Gamma^1_{21} & \Gamma^1_{22} \\ \Gamma^2_{21} & \Gamma^2_{22} \end{pmatrix}$$

$$[\nabla_1, \nabla_2] = \cancelto{0}{[\partial_1, \partial_2]} + [\partial_1, A_2] + [A_1, \partial_2] + [A_1, A_2]$$

$$\left[\frac{d}{dx}, f(x)\right] g(x) = \frac{d}{dx}(f \cdot g) - f \frac{d}{dx}(g)$$

$$= \frac{df}{dx} \cdot g + \cancel{f \frac{dg}{dx}} - \cancel{f \frac{dg}{dx}}$$

$$\left[\frac{d}{dx}, f\right] = \frac{df}{dx}$$

$$[\nabla_1, \nabla_2] = \partial_1 A_2 - \partial_2 A_1 + [A_1, A_2]$$

$$= \begin{pmatrix} \partial_1 \Gamma^1_{21} & \partial_1 \Gamma^1_{22} \\ \partial_1 \Gamma^2_{21} & \partial_1 \Gamma^2_{22} \end{pmatrix} - \begin{pmatrix} \partial_2 \Gamma^1_{11} & \partial_2 \Gamma^1_{12} \\ \partial_2 \Gamma^2_{11} & \partial_2 \Gamma^2_{12} \end{pmatrix}$$

$$+ \left[\begin{pmatrix} \Gamma^1_{11} & \Gamma^1_{12} \\ \Gamma^2_{11} & \Gamma^2_{12} \end{pmatrix}, \begin{pmatrix} \Gamma^1_{21} & \Gamma^1_{22} \\ \Gamma^2_{21} & \Gamma^2_{22} \end{pmatrix} \right]$$

$$= \begin{pmatrix} \sim & \sim \\ \begin{matrix} \partial_1 \Gamma^2_{21} - \partial_2 \Gamma^2_{11} \\ + \Gamma^2_{11}\Gamma^1_{21} + \Gamma^2_{12}\Gamma^2_{21} \\ - \Gamma^1_{11}\Gamma^2_{21} - \Gamma^2_{11}\Gamma^2_{22} \end{matrix} & \sim \end{pmatrix}$$

$$= -C_1 \sqrt{EG-F^2} \cdot \frac{1}{\sqrt{EG-F^2}} \begin{pmatrix} -F & -G \\ E & F \end{pmatrix}$$

$$-C_1 E = \partial_1 \Gamma_{21}^{2} - \partial_2 \Gamma_{11}^{2} + (\text{stuff})$$

$$-C_1 F = \sim$$

$$C_1 F = \sim$$

$$-C_1 G = \sim$$

$$\boxed{\left(\partial_1 A_2 - \partial_2 A_1 + [A_1, A_2]\right) = +C_1 \begin{pmatrix} F & G \\ -E & -F \end{pmatrix}}$$

Gauss equations.

$$(\nabla_i w)^j = \partial_i w^j + \sum_K \Gamma^j_{iK} w^K$$

$$\Gamma_i{}^j{}_K \qquad\qquad \Gamma^j_{iK}$$

$$A_1 = \begin{pmatrix} \Gamma_1{}^1{}_1 & \Gamma_1{}^1{}_2 \\ \Gamma_1{}^2{}_1 & \Gamma_1{}^2{}_2 \end{pmatrix}$$

In $n > 2$ dimensions.

$$A_i = \begin{array}{cc} & K \\ j & \begin{pmatrix} \quad \Gamma_i{}^j{}_K \quad \end{pmatrix} \end{array}$$

$$R_{ijK}{}^l = [\nabla_i, \nabla_j]_K{}^l \qquad \text{Riemann curvature tensor.}$$

$$R_{ijKl} = \sum_{l'} g_{ll'} R_{ijK}{}^{l'} = -R_{jiKl} = -R_{ijlK}$$