2nd fundamental form

$e_1 = \sigma_u$

$e_2 = \sigma_v$

$$\langle\langle v, w \rangle\rangle = -\vec{N}_{\vec{v}} \cdot \vec{w}$$

$$L = \langle\langle e_1, e_1 \rangle\rangle = \vec{\sigma}_{uu} \cdot \vec{N}$$

$$M = \langle\langle e_1, e_2 \rangle\rangle = \langle\langle e_2, e_2 \rangle\rangle = \vec{\sigma}_{uv} \cdot \vec{N}$$

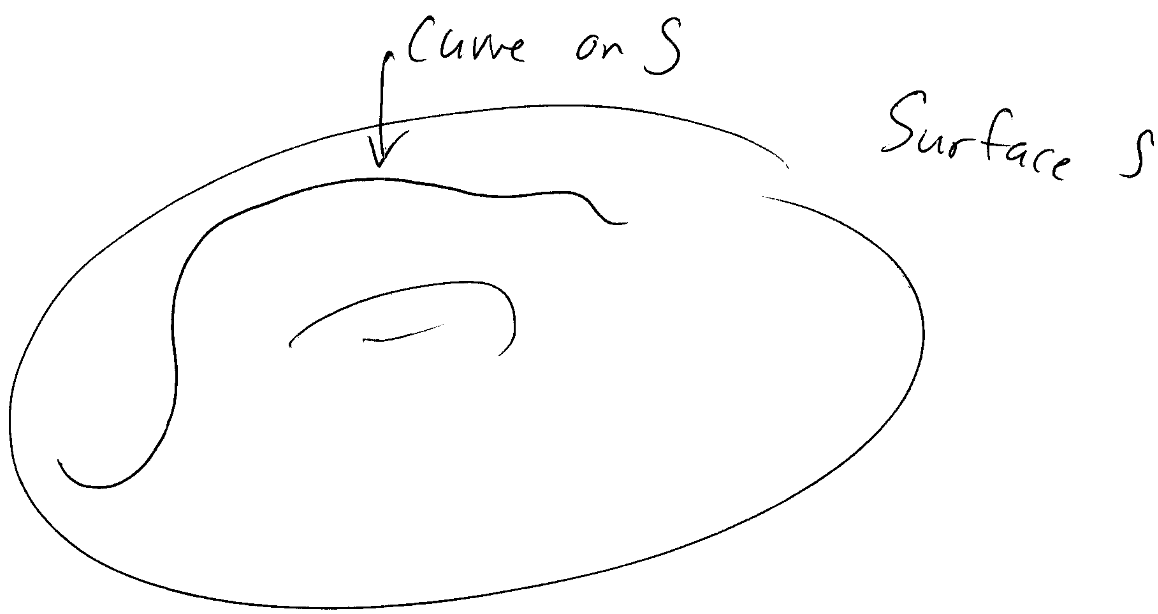
$$N = \langle\langle e_2, e_2 \rangle\rangle = \vec{\sigma}_{vv} \cdot \vec{N}$$

$$W(\vec{v}) = -\vec{N}_{\vec{v}}$$

$$= -dG$$

$$G(p) = \vec{N}_p(p)$$

$W : T_p S \to T_p S$



$$\gamma(t) = \sigma(u(t), v(t))$$

$$|\dot{\gamma}| = \text{constant} = 1$$

$$\ddot{\gamma} \perp \dot{\gamma}$$

$$\ddot{\gamma} = \underbrace{\kappa_n}_{\substack{\text{normal}\\ \text{curvature}}} \vec{N} + \underbrace{\kappa_g}_{\substack{\text{geodesic}\\ \text{curvature}}} (\vec{N} \times \dot{\gamma})$$

$$\kappa^2 = |\ddot{\gamma}|^2 = \kappa_n^2 + \kappa_g^2$$

$$\kappa_n = \kappa \cos\psi, \quad \kappa_g = \kappa \sin\psi$$

<u>Thm</u> If $|\dot{\gamma}| = 1$, $\quad K_n = \langle\langle \dot{\gamma}, \dot{\gamma} \rangle\rangle$

$$\dot{\gamma} = \dot{u}\,\sigma_u + \dot{v}\,\sigma_v$$

$$\ddot{\gamma} = \ddot{u}\,\sigma_u + \dot{u}\,\dot{\sigma}_u + \ddot{v}\,\sigma_v + \dot{v}\,\dot{\sigma}_v$$

$$= (\ddot{u}\,\sigma_u + \ddot{v}\,\sigma_v) + \dot{u}\,(\sigma_{uu}\dot{u} + \sigma_{uv}\dot{v}) + \dot{v}\,(\sigma_{uv}\dot{u} + \sigma_{vv}\dot{v})$$

$$\ddot{\gamma} \cdot \vec{N} = (\dot{u})^2 (\vec{\sigma}_{uu} \cdot \vec{N}) + 2\dot{u}\dot{v}\,(\vec{\sigma}_{uv} \cdot \vec{N}) + (\dot{v})^2 (\vec{\sigma}_{vv} \cdot \vec{N})$$

$$= L(\dot{u})^2 + 2M\,\dot{u}\dot{v} + N(\dot{v})^2 = (\dot{u}\ \ \dot{v}) \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}$$

$$= \langle\langle \dot{\gamma}, \dot{\gamma} \rangle\rangle$$

<u>Defn</u>: A <u>geodesic</u> is a curve with $K_g = 0$.

Covariant derivative along a curve

vector field $\vec{V}(t) \in T_{\gamma(t)} S$

$\nabla_\gamma \vec{V} =$ part of $\dot{\vec{V}}$ that is tangent to S.

$$= \dot{\vec{V}} - (\dot{\vec{V}} \cdot \vec{N})\vec{N}$$

~~Q: What~~ ~~is~~ $\nabla_\gamma \dot{\gamma} = \ddot{\gamma} - (\ddot{\gamma} \cdot \vec{N})\vec{\gamma}$

$$= K_g (\vec{N} \times \dot{\gamma})$$

A curve is a geodesic if $\nabla_\gamma \dot{\gamma} = 0$

Basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} = \{\vec{\sigma}_u, \vec{\sigma}_v, \vec{N}\}$.

for $i, j = 1, 2$.

$$\nabla_{e_i} e_j = \underbrace{\Gamma^1_{ij}} e_1 + \underbrace{\Gamma^2_{ij}} e_2$$

Christoffel symbols

$$\nabla_i e_j = \sum_k \Gamma^k_{ij} e_k$$

$\nabla_1 e_1 =$ tangential part of σ_{uu}

$$\sigma_{uu} = \underbrace{\Gamma^1_{11} e_1 + \Gamma^2_{11} e_2} + L \vec{N}$$

$$\sigma_{uu} \cdot \sigma_u = \Gamma^1_{11} E + \Gamma^2_{11} F +$$

$$\frac{1}{2} \partial_u (\sigma_u \cdot \sigma_u) = \Gamma^1_{11} E + \Gamma^2_{11} F$$

$$\Gamma^1_{11} E + \Gamma^2_{11} F = \frac{1}{2} E_u$$

$$\sigma_{uu} \cdot \sigma_v = \Gamma^1_{11} F + \Gamma^2_{11} G$$

$$\sigma_{uu}\cdot\sigma_v = \partial_u(\sigma_u\cdot\sigma_v) - \sigma_u\cdot\sigma_{uv}$$

$$= \partial_u(\sigma_u\cdot\sigma_v) - \frac{1}{2}\partial_v(\sigma_u\cdot\sigma_u)$$

$$= F_u - \frac{1}{2}E_v$$

$$E\,\Gamma_{11}^{1} + F\,\Gamma_{11}^{2} = \frac{1}{2}E_u$$

$$F\,\Gamma_{11}^{1} + G\,\Gamma_{11}^{2} = F_u - \frac{1}{2}E_v$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \Gamma_{11}^{1} \\ \Gamma_{11}^{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}E_u \\ F_u - \frac{1}{2}E_v \end{pmatrix}$$

$$\begin{pmatrix} \Gamma_{11}^{1} \\ \Gamma_{11}^{2} \end{pmatrix} = \frac{1}{EG-F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} \frac{1}{2}E_u \\ F_u - \frac{1}{2}E_v \end{pmatrix}$$

$$\begin{pmatrix} \Gamma_{21}^{1} \\ \Gamma_{21}^{2} \end{pmatrix} = \begin{pmatrix} \Gamma_{12}^{1} \\ \Gamma_{12}^{2} \end{pmatrix} = \frac{1}{EG-F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} E_v \\ G_u \end{pmatrix}$$

$$\begin{pmatrix} \Gamma_{22}^{1} \\ \Gamma_{22}^{2} \end{pmatrix} = \frac{1}{EG-F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} F_v - \frac{1}{2}G_u \\ \frac{1}{2}G_v \end{pmatrix}$$

$$\vec{W} = w^1 \vec{e}_1 + w^2 \vec{e}_2$$

$$\dot{\vec{w}} = \underline{\dot{w}^1 e_1} + \dot{w}^2 e_2 + w^1 \dot{e}_1 + w^2 \dot{e}_2$$

horizontal part of $\dot{\vec{w}} = \nabla_\gamma \vec{w}$

$\cancel{= \dot{w}^1 e_1 + \dot{w}^2 e_2 + w^1 (\ldots) e_1}$ ~~horizon~~

$$\dot{e}_1 = \dot{u}\, \underline{\sigma_{uu}} + \dot{v}\, \underline{\sigma_{uv}}$$

horizontal part of $\dot{e}_1 = \dot{u}\left(\Gamma^1_{11} e_1 + \Gamma^2_{11} e_2\right) + \dot{v}\left(\Gamma^1_{12} e_1 + \Gamma^2_{12} e_2\right)$

horizontal part of $\dot{e}_2 = \dot{u}\left(\Gamma^1_{21} e_1 + \Gamma^2_{21} e_2\right) + \dot{v}\left(\Gamma^1_{22} e_1 + \Gamma^2_{22} e_2\right)$

$$\nabla_\gamma \vec{w} = \left(\dot{w}^1 + w^1 \dot{u}\, \Gamma^1_{11} + \dot{v} w^1 \Gamma^1_{12} + w^2 \dot{u}\, \Gamma^1_{21} + w^2 \dot{v}\, \Gamma^1_{22}\right) e_1$$

$$\left(\dot{w}^2 + \text{4 more terms}\right) e_2$$

$$\left(\nabla_\gamma \vec{w}\right)^i = \dot{w}^i + \sum_{jk} \Gamma^i_{jk}\, w^j (\dot{\gamma})^k$$

W is self-parallel along γ if $\nabla_\gamma \vec{w} = 0$

$$\dot{w}^i = -\sum_{jk} \Gamma^i_{jk}\, w^j (\dot{\gamma})^k$$